

ON FIXED POINT PLANAR ALGEBRAS

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ABSTRACT. To a weighted graph can be associated a bipartite graph planar algebra \mathcal{P} . We construct and study the symmetric enveloping inclusion of \mathcal{P} . We show that this construction is equivariant with respect to the automorphism group of \mathcal{P} . We consider subgroups G of the automorphism of \mathcal{P} such that the G -fixed point space \mathcal{P}^G is a subfactor planar algebra. As an application we show that if G is amenable, then \mathcal{P}^G is amenable as a subfactor planar algebra. We define the notions of a cocycle action of a Hecke pair on a tracial von Neumann algebra and the corresponding crossed product. We show that a large class of symmetric enveloping inclusions of subfactor planar algebras can be described by such a crossed product.

INTRODUCTION AND MAIN RESULTS

The theory of subfactors has been initiated by Jones [13]. Given a subfactor (an extremal unital inclusion of factors of type II_1 with finite index), Jones associated a combinatorial invariant called the standard invariant. It has been axiomatized in the finite depth case as a paragroup by Ocneanu [22]. Then, it has been axiomatized in the general case as a λ -lattice by Popa [29] and as a subfactor planar algebra by Jones [12]. The reconstruction theorem of Popa shows that any λ -lattice is the standard invariant of a subfactor, see [26, 29, 31].

Popa studied analytic properties of subfactors and among other defined the notion of amenability in this context [27]. He gave many characterizations of amenability and defined it for λ -lattices and thus for subfactor planar algebras. He constructed the so-called symmetric enveloping inclusion $T \subset S$ associated to a subfactor $N \subset M$ [28] which extends constructions due to Ocneanu, and Longo and Rehren [22, 19]. The symmetric enveloping inclusion is a subfactor of type II_1 and has finite index if and only if $N \subset M$ has finite depth. Let \mathcal{P} be the subfactor planar algebra of a subfactor $N \subset M$. Popa proved in [30] that \mathcal{P} is amenable if and only if the symmetric enveloping inclusion of $N \subset M$ is co-amenable, see Section 1.3.

Given a subfactor planar algebra \mathcal{Q} , one can construct a tower of II_1 factors $M_0 \subset M_1 \subset M_2 \subset \dots$ and a sequence of symmetric enveloping inclusions $T_k \subset S_k, k \geq 0$ [11, 8]. It has been proven that the standard invariant of $M_0 \subset M_1$ is the subfactor planar algebra \mathcal{Q} and the symmetric enveloping inclusion of $M_{k-1} \subset M_k$ is isomorphic to the inclusion $T_k \subset S_k$ for any $k \geq 1$. We call $T_0 \subset S_0$ the symmetric enveloping inclusion associated to \mathcal{Q} .

Given a bipartite graph Γ and a weight μ on its edges satisfying certain assumptions we can construct a planar algebra \mathcal{P} called a bipartite graph planar algebra [14, 7]. See also [21]. The automorphism group $\text{Aut}(\mathcal{P})$ of \mathcal{P} is isomorphic to a semi-direct product $U(\mathcal{P}_1^+) \rtimes \text{Aut}(\Gamma, \mu)$ where $\text{Aut}(\Gamma, \mu)$ is the group of automorphisms of Γ that preserves the weight μ and $U(\mathcal{P}_1^+)$ is the unitary group of the 1-box space \mathcal{P}_1^+ . The group $\text{Aut}(\mathcal{P})$

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acts on the vertices $V = V^+ \cup V^-$ of Γ where $U(\mathcal{P}_1^+)$ acts trivially. If $G < \text{Aut}(\mathcal{P})$ is any subgroup, then the fixed point space \mathcal{P}^G is a planar algebra. If G acts transitively on the even and odd vertices of Γ , then \mathcal{P}^G satisfies automatically all the axioms of a subfactor planar algebra except sphericity. In particular we obtain that \mathcal{P}^G is non-degenerate which is usually the hardest axiom to check for a subfactor planar algebra, see Section 1 for definitions. Note that starting from a bipartite graph Γ and a vertex-transitive group action $G \curvearrowright \Gamma$, there exists a unique weight μ such that we can associate to (Γ, μ) a bipartite graph planar algebra \mathcal{P} and such that the fixed point space \mathcal{P}^G is a subfactor planar algebra, see [2, Proposition 2.5].

In this article, we consider bipartite graph planar algebras \mathcal{P} and their planar subalgebras $\mathcal{Q} \subset \mathcal{P}$. We investigate the structure of \mathcal{Q} , its symmetric enveloping inclusion, and the notion of amenability when \mathcal{Q} is a subfactor planar algebra embedded in \mathcal{P} . We start by proving that \mathcal{Q} is non-amenable if $\|\Gamma\| < \delta$, where δ is the modulus of \mathcal{Q} . The proof is elementary and is independent from the rest of the paper. We extend the construction of [11, 8] and associate to a bipartite graph planar algebra \mathcal{P} a tower of von Neumann algebras $M_0 \subset M_1 \subset M_2 \subset \dots$ and a sequence of inclusions $T_k \subset S_k, k \geq 0$. Note that the construction of the tower was already explained in [11, Section 4]. We prove that M_i and T_k are von Neumann algebras of type II_1 with atomic centers and that S_k is a factor of type II for any $i, k \geq 0$. We show that the automorphism group $\text{Aut}(\mathcal{P})$ of \mathcal{P} acts in a minimal way on those von Neumann algebras and show that the constructions of M_i and S_k behaves as expected with respect to inclusions of planar algebras and group actions. In particular, if $G < \text{Aut}(\mathcal{P})$ is a subgroup such that \mathcal{P}^G is a subfactor planar algebra, then the inclusion of fixed point spaces $M_0^G \subset M_1^G$ is a subfactor with standard invariant isomorphic to \mathcal{P}^G and the symmetric enveloping inclusion of this subfactor (resp. of \mathcal{P}^G) is isomorphic to $M_1^G \vee (M_1^{\text{op}})^G \subset S_1^G$ (resp. $M_0^G \vee (M_0^{\text{op}})^G \subset S_0^G$). As an application we prove the following theorem.

Theorem A. *Consider a bipartite graph planar algebra \mathcal{P} over a weighted graph (Γ, μ) and a subgroup $G < \text{Aut}(\mathcal{P})$ such that \mathcal{P}^G is a subfactor planar algebra. If the group G is amenable (as closed subgroup or a countable discrete group), then the subfactor planar algebra \mathcal{P}^G is amenable.*

Note that a more general framework has been studied independently by Arano and Vaes from which this theorem follows [2]. Recall that a Hecke pair (G, H) is an inclusion of groups $H < G$ which is almost normal, i.e. for any $g \in G$ the group $H \cap gHg^{-1}$ has finite index inside H and gHg^{-1} . We define the notion of a cocycle action of a Hecke pair on a tracial von Neumann algebra and the corresponding twisted crossed product von Neumann algebra. Note that it has been considered already in the framework of ordinary action on C^* -algebras by Palma, see [23, 24]. We show that if μ is constant on the set of even edges and $G < \text{Aut}(\mathcal{P})$ is a subgroup such that \mathcal{P}^G is a subfactor planar algebra, then the symmetric enveloping inclusion of \mathcal{P}^G can be described in the following way.

Theorem B. *Consider a bipartite graph planar algebra \mathcal{P} over a weighted graph (Γ, μ) such that μ is constant on the set of even edges and a subgroup $G < \text{Aut}(\mathcal{P})$. Assume that \mathcal{P}^G is a subfactor planar algebra. Denote by G_o the subgroup of G that fixes an even vertex o of Γ . There exists a II_1 factor A and a cocycle action of the Hecke pair (G, G_o) on $A \bar{\otimes} A^{\text{op}}$ such that G_o acts on A and such that the symmetric enveloping inclusion of \mathcal{P}^G is isomorphic to the inclusion $A^{G_o} \bar{\otimes} (A^{\text{op}})^{G_o} \subset vN[A \bar{\otimes} A^{\text{op}}; G, G_o]$, where $vN[A \bar{\otimes} A^{\text{op}}; G, G_o]$ is a twisted crossed product of $A \bar{\otimes} A^{\text{op}}$ by the Hecke pair (G, G_o) .*

See Theorem 4.5 for a slightly more precise statement. Many symmetric enveloping inclusion of subfactors can be described in that way including diagonal and Bisch-Haagerup subfactors, see Section 4.3. This theorem can be interpreted as an extension of a theorem of Popa which shows that the symmetric enveloping algebra S of a diagonal subfactor is the twisted crossed product of a von Neumann algebra by a group [27, Section 3].

1. PRELIMINARIES AND A CRITERION OF NON-AMENABILITY

A planar algebra is a collection of complex $*$ -algebras $\mathcal{P} = (\mathcal{P}_n^\pm : n \geq 0)$ on which the operad of shaded planar tangles acts. See [12, 15] for more details. We follow similar conventions to [8] for drawing a shaded planar tangle. We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We do not draw the outside box and will omit unnecessary decorations. The left and right traces of a planar algebra are the maps $\tau_l : \mathcal{P}_n^\pm \longrightarrow \mathcal{P}_0^\mp$ and $\tau_r : \mathcal{P}_n^\pm \longrightarrow \mathcal{P}_0^\pm$ defined for any $n \geq 0$ such that

$$\tau_l(x) = \boxed{x} \text{ and } \tau_r(x) = \boxed{x} \text{ for any } x \in \mathcal{P}_n^\pm.$$

Suppose that $\mathcal{P}_0^\pm = \mathbb{C}$. The planar algebra is called spherical if the two traces agree on each element of \mathcal{P} . We say that \mathcal{P} is non-degenerate if the sesquilinear forms $(x, y) \mapsto \tau_l(xy^*)$ and $(x, y) \mapsto \tau_r(xy^*)$ are positive definite. A subfactor planar algebra is a planar algebra such that each space \mathcal{P}_n^\pm is finite dimensional, $\mathcal{P}_0^\pm = \mathbb{C}$, \mathcal{P} is spherical, and is non-degenerate. The modulus of a subfactor planar algebra is the value of a closed loop. The index of a subfactor planar algebra is the square of its modulus. Note that all the subfactors considered in this article are extremal. This condition corresponds to the sphericity of their associated subfactor planar algebra. ‘

1.1. Bipartite graph planar algebras. We refer to [14] and [7] for more details about bipartite graph planar algebras of finite and infinite graphs respectively. Let Γ be a countable locally finite undirected connected bipartite graph that can have multiple edges between two vertices. Denote by $V = V^+ \sqcup V^-$ its set of vertices, where V^+ and V^- are the set of even and odd vertices respectively. We consider the associated symmetric oriented graph obtained by doubling each edge of Γ into a pair of oppositely oriented edges. If a is a path, then we denote by $\bar{a}, s(a), t(a)$ its associated opposite edge, its source, and its target respectively.

We put $C_0^\pm = V^\pm$, $C_n^\pm, n \geq 1$ the set of paths in Γ of length n that start in V^\pm , $C_*^\pm = \bigcup_{n \geq 0} C_n^\pm$. Consider the Hilbert space $\ell^2(C_n^\pm)$ with orthonormal basis indexed by C_n^\pm . Let $B(\ell^2(C_n^\pm))$ be the space of bounded linear operators on $\ell^2(C_n^\pm)$ with the standard system of matrix units $\{e_{a,b} : a, b \in C_n^\pm\}$. Define the von Neumann subalgebra $\mathcal{P}_n^\pm \subset B(\ell^2(C_n^\pm))$ generated by the operators $e_{a,b}$ where a, b are in C_n^\pm and have the same source and target. We denote by $ST_n^\pm, n \geq 0$ the set of couples (a, b) where $a, b \in C_n^\pm, s(a) = s(b)$, and $t(a) = t(b)$. We put $ST_*^\pm = \bigcup_{n \geq 0} ST_n^\pm$. Observe that the von Neumann algebra \mathcal{P}_0^\pm is isomorphic to the abelian atomic von Neumann algebra of bounded functions $\ell^\infty(V^\pm)$. We identify \mathcal{P}_0^\pm with $\ell^\infty(V^\pm)$. We denote by $\{e_v : v \in V^\pm\}$ the set of minimal projections of \mathcal{P}_0^\pm .

Assume there exists a strictly positive map $\mu : C_1 \longrightarrow \mathbb{R}_+^*$ called a weight satisfying that for any paths $a = a_1 \cdots a_n, b = b_1 \cdots b_l$ of length $n, l \geq 1$ such that $s(a_1) = s(b_1), t(a_n) =$

$t(b_l)$ we have that

$$(1) \quad \mu(a_1) \cdots \mu(a_n) = \mu(b_1) \cdots \mu(b_l) \text{ and}$$

$$(2) \quad \text{there exists } \delta > 0 \text{ such that } \sum_{c \in C_1: s(c)=v} \mu(c) = \delta \text{ for any } v \in V.$$

Such a pair (Γ, μ) is called a weighted graph with modulus δ or simply a weighted graph. Note that a weaker notion has been introduced and studied in [9] where the first assumption is replaced by $\mu(a)\mu(\bar{a}) = 1$ for any edge a where \bar{a} denotes the opposite edge of a .

Recall that the degree $\deg(v)$ of a vertex v is the number of edges with source v and the degree $\deg(\Gamma)$ of the graph Γ is define as $\sup_{v \in V} \deg(v)$. The existence of the weight μ implies that $\deg(\Gamma) \leq \delta^2$. Indeed, by (2), we have that $\mu(a) \leq \delta$ for any $a \in C_1$. The equality (1) and the inequality of above implies that $\delta^{-1} \leq \mu(a) \leq \delta$ for any $a \in C_1$. By using (2) again, we obtain that $\delta = \sum_{a \in C_1: s(a)=v} \mu(a) \geq \deg(v)\delta^{-1}$ for any $v \in V$. Therefore, $\deg(\Gamma) \leq \delta^2$.

We extend μ on the set of all paths of Γ by putting $\mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n)$ where $a_1 \cdots a_n$ is the concatenation path of n edges a_1, \dots, a_n . Up to multiplication by a strictly positive real number, there exists a unique map $\mu_V : V \rightarrow \mathbb{R}_+^*$ such that $\mu(a) = \mu_V(t(a))/\mu_V(s(a))$ for any $a \in C_1$. Indeed, let us fix a vertex $o \in V$ and put $\mu_V(o) = 1$ and $\mu_V(v) = \mu(a)$ where a is any path such that $s(a) = o$ and $t(a) = v$. Since Γ is assumed to be connected, there is always a path from o to v . Condition (1) assures that μ_V is well defined. Suppose there is another weight ν satisfying that $\mu(a) = \nu(t(a))/\nu(s(a))$ for any $a \in C_1$. We can see that $\nu(v) = \mu_V(v)\nu(o)$ for any $v \in V$. Hence, μ_V is unique up to a multiplication by a strictly positive real number. This map satisfies that $A_\Gamma(\mu_V) = \delta\mu_V$, where A_Γ is the adjacency matrix of the graph Γ . Following [14, 7], the data of (Γ, μ_V) allows us to define a planar algebra structure on $\mathcal{P}_\Gamma = \mathcal{P} = (\mathcal{P}_n^\pm : n \geq 0)$. Note that this planar algebra structure only depends on Γ and μ . We say that \mathcal{P} is the bipartite graph planar algebra associated to the weighted graph (Γ, μ) .

The bipartite graph planar algebra \mathcal{P} is known to satisfy the identity

$$(3) \quad \tau_l(e_{a,b}) = \delta_{a,b} e_{t(a)} \mu(\bar{a}) \text{ and } \tau_r(e_{a,b}) = \delta_{a,b} e_{s(a)} \mu(a)$$

for any $(a,b) \in ST_*^\pm$, where $\delta_{a,b}$ is the Kronecker symbol.

Note that condition (1) on μ assures the existence of μ_V and implies that τ_l and τ_r are tracial. More precisely, consider the collection of $*$ -algebras $(\mathcal{P}_n^\pm : n \geq 0)$ and the linear functional $\tau_l, \tau_r : \mathcal{P}_n^\pm \rightarrow \mathcal{P}_0, n \geq 0$ defined on the system of matrix units by equation (3). Then τ_l, τ_r are tracial if and only if $\mu(a) = \mu(b)$ for any $(a,b) \in ST_*^\pm$. Indeed, suppose that τ_r is tracial. If $(a,b) \in ST_*^\pm$, then $\tau_r(e_{a,b}e_{b,a}) = e_{s(a)}\mu(a) = \tau_r(e_{b,a}e_{a,b}) = e_{s(a)}\mu(b)$. Conversely, suppose that $\mu(a) = \mu(b)$ for any $(a,b) \in ST_*^\pm$. Consider $n \geq 1, x = \sum_{(a,b) \in ST_n^\pm} x_{a,b} e_{a,b}, y = \sum_{(a,b) \in ST_n^\pm} y_{a,b} e_{a,b}$. We have that $\tau_r(xy) = \sum_{(a,b) \in ST_n^\pm} x_{a,b} y_{b,a} \mu(a) e_{s(a)}$ and $\tau_r(yx) = \sum_{(a,b) \in ST_n^\pm} y_{b,a} x_{a,b} \mu(b) e_{s(b)} = \sum_{(a,b) \in ST_n^\pm} x_{a,b} y_{b,a} \mu(a) e_{s(a)} = \tau_r(xy)$. A similar proof shows that τ_l is also tracial.

1.2. Amenability. All groups that we consider are either countable discrete or locally compact second countable. This implies that any quotient space by a closed subgroup is countable. Recall that a locally compact group G is amenable if and only if for any affine

action of G on a compact convex subset of a locally convex topological vector space there exists a G -fixed point. Eymard defined and studied amenability of inclusion of groups [10]. A closed subgroup $H < G$ is co-amenable if and only if for any affine action of G on a compact convex subset of a locally convex topological vector space that admits a H -fixed point there exists a G -fixed point. Remark that if G is an amenable group, then any closed subgroup $H < G$ is co-amenable. There are alternative definitions of co-amenable when the subgroup $H < G$ is almost normal, i.e. for any $g \in G$ the group $H \cap gHg^{-1}$ has finite index inside H and gHg^{-1} . It has been proven in [1, Theorem 3.8] that a closed almost normal subgroup $H < G$ is co-amenable if and only if there exists a sequence of positive definite H -bi-invariant maps $f_n : G \rightarrow \mathbb{C}, n \geq 0$ with support contained in a finite union of right H -cosets and that converges pointwise to 1. We will use later this characterization.

Recall that if $\mathcal{N} \subset \mathcal{M}$ is an inclusion of II_1 factors and $L^2(\mathcal{M}, \tau)$ is the GNS Hilbert space associated to the unique normal faithful tracial state of \mathcal{M} we say that a \mathcal{N} -bimodule $K \subset L^2(\mathcal{M}, \tau)$ is bifinite if there exists a finite subset $F \subset L^2(\mathcal{M}, \tau)$ such that K is contained in the closure of $\text{Span}(x \cdot \xi; x \in \mathcal{N}, \xi \in F)$ and in the closure of $\text{Span}(\xi \cdot x : x \in \mathcal{N}, \xi \in F)$. Similarly to the group case, we say that an inclusion of II_1 factors $\mathcal{N} \subset \mathcal{M}$ is co-amenable if there exists a sequence of completely positive \mathcal{N} -bimodular maps $\phi_n : \mathcal{M} \rightarrow \mathcal{M}$ such that their image is a bifinite \mathcal{N} -bimodule and $\lim_{n \rightarrow \infty} \|\phi_n(x) - x\|_2 = 0$ for any $x \in \mathcal{M}$.

Let \mathcal{Q} be a subfactor planar algebra with loop parameter δ . Popa defined amenability for subfactors and λ -lattices [27]. In planar algebraic terms, \mathcal{Q} is amenable if $\|\Gamma(\mathcal{Q})\| = \delta$, where $\Gamma(\mathcal{Q})$ is the principal graph of \mathcal{Q} deduced from the Bratteli diagram of the tower of finite dimensional semi-simple $*$ -algebras $\mathcal{Q}_0^+ \subset \mathcal{Q}_1^+ \subset \dots$ and where $\|\Gamma(\mathcal{Q})\|$ is the operator norm of the adjacency matrix of Γ acting on the ℓ^2 -space of the vertices of $\Gamma(\mathcal{Q})$. If \mathcal{Q} is a subfactor planar algebra, then we can associate a sequence of symmetric enveloping inclusions $T_k \subset S_k, k \geq 0$ as constructed in [8]. Those are inclusions of II_1 factors. It can be shown that $T_1 \subset S_1$ is isomorphic to the symmetric enveloping inclusion of a subfactor having its subfactor planar algebra isomorphic to \mathcal{Q} , see [8, Theorem 3.3]. Popa proved that \mathcal{Q} is amenable if and only if the symmetric enveloping inclusion $T_1 \subset S_1$ is co-amenable [30, Theorem 5.3]. Furthermore, up to a compression and finite index inclusions we have that $T_1 \subset S_1$ is isomorphic to $T_0 \subset S_0$ [6, Lemma 4.3]. Therefore, \mathcal{Q} is amenable if and only if $T_0 \subset S_0$ is co-amenable. We will later use this characterization of amenability of subfactor planar algebras.

1.3. A criterion of amenability for subfactor planar algebras. We provide a criteria of non-amenability for a subfactor planar algebra embedded in a bipartite graph planar algebra. The proof of this criteria is elementary and is interesting by its own.

Theorem 1.1. *Let (Γ, μ, δ) be a weighted graph with a modulus such that $\|\Gamma\| < \delta$. Consider its associated bipartite graph planar algebra \mathcal{P} . If \mathcal{Q} is a subfactor planar algebra that embeds inside \mathcal{P} , then \mathcal{Q} is non-amenable.*

Proof. Let $\Gamma, \mu, \delta, \mathcal{Q}$ be as above. Denote by $\Gamma(\mathcal{Q})$ the principal graph of \mathcal{Q} . It is sufficient to show that $\|\Gamma(\mathcal{Q})\| \leq \|\Gamma\|$ since \mathcal{Q} and \mathcal{P} have the same modulus δ . Let $\Gamma(\mathcal{Q})_n$ be the Bratteli diagram of the inclusion $\mathcal{Q}_n^+ \subset \mathcal{Q}_{n+1}^+$ for $n \geq 0$. We have the equality

$$(4) \quad \|\Gamma(\mathcal{Q})\| = \lim_{n \rightarrow \infty} \|\Gamma(\mathcal{Q})_n\|.$$

Let Δ_n be the Bratteli diagram of the inclusion $\mathcal{P}_n^+ \subset \mathcal{P}_{n+1}^+$. This graph is equal to the disjoint union of graphs $\Delta_n = \bigsqcup_{v \in V^+} \Delta_n(v)$, where $\Delta_n(v)$ is the subgraph of Γ with vertices $V_n(v)^\pm = \{w \in V^\pm : d(v, w) \leq n+1\}$ and edges the one of Γ . Since Δ_n is a subgraph of Γ we have that

$$(5) \quad \|\Delta_n\| \leq \|\Gamma\|, \text{ for any } n \geq 0.$$

The planar algebras \mathcal{Q} is contained inside \mathcal{P} . Fix $n \geq 0$ and consider the following square of inclusions

$$(6) \quad \begin{array}{ccc} \mathcal{P}_n^+ & \subset & \mathcal{P}_{n+1}^+ \\ \cup & & \cup \\ \mathcal{Q}_n^+ & \subset & \mathcal{Q}_{n+1}^+ \end{array}.$$

Consider the normalized tracial operator

$$tr_m : \mathcal{P}_m^+ \longrightarrow \mathcal{P}_0^+, x \longmapsto \frac{1}{\delta^m} \boxed{x} \text{ for any } m \geq 0.$$

Observe that the restriction of tr_{n+1} to \mathcal{P}_n^+ is equal to tr_n . We equipped \mathcal{P}_n^+ , \mathcal{Q}_n^+ , and \mathcal{Q}_{n+1}^+ with the corresponding restrictions of tr_{n+1} . Note that

$$E_{\mathcal{P}} : \mathcal{P}_{n+1}^+ \longrightarrow \mathcal{P}_n^+, x \longmapsto \frac{1}{\delta} \boxed{x}$$

is a faithful conditional expectation satisfying $tr_n \circ E_{\mathcal{P}} = tr_{n+1}$. The restriction of $E_{\mathcal{P}}$ to \mathcal{Q}_{n+1}^+ is a trace-preserving conditional expectation from \mathcal{Q}_{n+1}^+ onto \mathcal{Q}_n^+ . Therefore, the square of inclusion (6) is a commuting square of inclusions with respect to the conditional expectations $E_{\mathcal{P}}$ and $E_{\mathcal{P}}|_{\mathcal{Q}_{n+1}^+}$. Let L_n be the von Neumann algebra generated by \mathcal{P}_n^+ and \mathcal{Q}_{n+1}^+ inside \mathcal{P}_{n+1}^+ . Denote by Λ_n the Bratteli diagram of the inclusion $\mathcal{P}_n^+ \subset L_n$. The inclusions $\mathcal{Q}_n^+ \subset \mathcal{Q}_{n+1}^+ \subset L_n$ and $\mathcal{P}_n^+ \subset L_n$ form a non-degenerate commuting square. Therefore, $\|\Gamma(\mathcal{Q})_n\| = \|\Lambda_n\|$ by [18, Remark 5.3.5]. This implies that

$$(7) \quad \|\Gamma(\mathcal{Q})_n\| \leq \|\Delta_n\|, \text{ for any } n \geq 0.$$

We obtain the result by combining (4), (5), and (7). \square

Remark 1.2. The embedding theorem of [16, 21] implies the following. If \mathcal{Q} is a non-amenable subfactor planar algebra, then there exists a bipartite graph planar algebra \mathcal{P} associated to a weighted graph with a modulus (Γ, μ, δ) such that $\|\Gamma\| < \delta$ and such that \mathcal{Q} embeds inside \mathcal{P} . Therefore, a subfactor planar algebra \mathcal{Q} is non-amenable if and only if there exists a weighted graph with a modulus (Γ, μ, δ) such that $\|\Gamma\| < \delta$ and such that \mathcal{Q} embeds in the bipartite graph planar algebra associated to (Γ, μ, δ) .

2. CONSTRUCTION OF VON NEUMANN ALGEBRAS

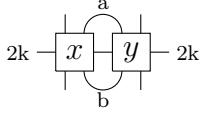
2.1. Von Neumann algebras associated to a weighted graph with a modulus.

We fix a weighted graph (Γ, μ) with modulus δ and consider the bipartite graph planar algebra $\mathcal{P} = \mathcal{P}_\Gamma$. We generalize the construction of [11, 17] and [8] and provide a tower of von Neumann algebras and a family of symmetric enveloping inclusions associated to \mathcal{P} . Let $k \geq 0$ be a natural number and ϵ the sign $+$ is k is even and $-$ if k is odd. For any $n, m \geq 0$, let $D_k(n, m)$ be a copy of the vector space \mathcal{P}_{n+m+2k}^+ .

Consider the direct sum


$$Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} := \bigoplus_{n,m \geq 0} D_k(n, m)$$

that we equipped with the Bacher product:


$$xy = \sum_{a=0}^{\min(2n, 2i)} \sum_{b=0}^{\min(2m, 2j)} 2k \cdot \text{diagram}$$


where $x \in D_k(n, m)$, $y \in D_k(i, j)$, and there are $2k$ horizontal strands in the middle.

Let $\dagger : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ be the anti-linear involution that sends $D_k(n, m)$ to itself and satisfies

$$x^\dagger = - \text{diagram}^{\$}, \text{ for any } x \in D_k(n, m).$$


Consider the linear map $E : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow \mathcal{P}_0^\epsilon$ which sends $x \in D_k(0, 0)$ to

$$\delta^{-2k} \text{diagram}$$


and 0 to any element in $D_k(n, m)$ with $(n, m) \neq (0, 0)$. The vector space $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ endowed with those operation is an associative $*$ -algebra with a faithful positive linear map to \mathcal{P}_0^ϵ . We fix a weight $\mu_V : V \rightarrow \mathbb{R}_+$ that satisfies that $\mu(a) = \mu_V(t(a))/\mu_V(s(a))$ for any $a \in C_1$. Consider the normal faithful semi-finite weight τ_V satisfying $\tau_V(e_v) = \mu_V(v)^2$ for any $v \in V$. Denote by $H_k(n, m)$ the Hilbert space equal to the completion of the pre-Hilbert space $\{x \in D_k(n, m) : \tau_V \circ E(xx^\dagger) < \infty\}$ equipped with the inner product $\langle x, y \rangle = \tau_V \circ E(xy^\dagger)$, for any $n, m \geq 0$. Let $H_k = \bigoplus_{n,m \geq 0} H_k(n, m)$ be the direct sum of those Hilbert spaces. Consider $\pi_k : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow \mathcal{L}(K_k)$, the left regular representation of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$, where $K_k = H_k \cap Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ and $\mathcal{L}(K_k)$ is the algebra of linear maps from K_k to K_k .

Proposition 2.1. *For any $x \in Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$, the linear map $\pi_k(x)$ defines a bounded operator on H_k .*

Proof. The proof follows the same ideas as [17, Theorem 3.3]. \square

Denote by S_k the von Neumann algebra generated by the image of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ inside $B(H_k)$. We continue to denote by π_k the representation of S_k on H_k . Let $Gr_k \mathcal{P}$ be the subalgebra of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ generated by $\bigcup_{n \geq 0} D_k(n, 0)$. We identify the opposite algebra of $Gr_k \mathcal{P}$ with the subalgebra of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ generated by $\bigcup_{m \geq 0} D_k(0, m)$ that we denote by $Gr_k \mathcal{P}^{\text{op}}$. Let $\mathbb{C}[V^\epsilon] \subset \ell^\infty(V^\epsilon)$ be the subalgebra generated by the set of projections $\{e_v, v \in V^\epsilon\}$. The abelian algebra $\mathbb{C}[V^\epsilon]$ is contained in the center of $Gr_k \mathcal{P}$ and $Gr_k \mathcal{P}^{\text{op}}$. Further, the subalgebra of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ generated by $Gr_k \mathcal{P}$ and $Gr_k \mathcal{P}^{\text{op}}$ is isomorphic to the tensor product of $Gr_k \mathcal{P}$ and $Gr_k \mathcal{P}^{\text{op}}$ over $\mathbb{C}[V^\epsilon]$. We denote this subalgebra by

$$Gr_k \mathcal{P} \otimes_{\mathbb{C}[V^\epsilon]} Gr_k \mathcal{P}^{\text{op}}.$$

Denote by T_k and M_k the von Neumann subalgebras of S_k generated by $Gr_k \mathcal{P} \otimes_{\mathbb{C}[V^\epsilon]} Gr_k \mathcal{P}^{\text{op}}$ and $Gr_k \mathcal{P}$ respectively. The von Neumann algebra generated by $Gr_k \mathcal{P}^{\text{op}}$ is isomorphic to the opposite von Neumann algebra of M_k . We denote it by M_k^{op} . There exists an inclusion of $*$ -algebras $i_k : Gr_k \mathcal{P} \otimes_{\mathbb{C}[V^\epsilon]} Gr_k \mathcal{P}^{\text{op}} \rightarrow Gr_{k+1} \mathcal{P} \otimes_{\mathbb{C}[V^\epsilon]} Gr_{k+1} \mathcal{P}^{\text{op}}$ which

consists of adding two horizontal lines in the middle and dividing by δ^2 , where ε is $+$ if k is odd and $-$ if k is even. This inclusion induces unital embeddings from T_k into T_{k+1} , M_k into M_{k+1} , and M_k^{op} into M_{k+1}^{op} .

2.2. Properties associated to those von Neumann algebras. For any $n, m \geq 0$, denote by $j_k(n, m)$ the canonical embedding of \mathcal{P}_{n+m+2k}^+ into $D_k(n, m)$ viewed as a subspace of S_k . For any vertex $v \in V^\varepsilon$, consider the projection

$$p_v = \frac{\text{---}k\text{---}}{\text{---}k\text{---}} \boxed{e_v} = j_k(0, 0)(e_v) \in S_k.$$

This projection belongs to $M_k \cap M_k^{\text{op}}$ and commutes with T_k . Denote by $T_k(v)$, $M_k(v)$, $M_k(v)^{\text{op}}$ the corners $T_k p_v$, $M_k p_v$, and $M_k^{\text{op}} p_v$ respectively.

Proposition 2.2. *The following assertions are true.*

- (1) *The von Neumann algebra $T_k(v)$ is isomorphic to the tensor product $M_k(v) \bar{\otimes} M_k(v)^{\text{op}}$ and $M_k(v)$ is a II_1 factor, for any $v \in V^\varepsilon$.*
- (2) *The corner $p_v S_k p_v$ is equal to $T_k(v)$ for any $v \in V^\varepsilon$.*
- (3) *The relative commutant $T'_k \cap S_k$ is equal to the center $Z(T_k)$ of T_k and is isomorphic to $\mathcal{P}_0^\varepsilon$. The set of minimal central projections of T_k is $\{p_v : v \in V^\varepsilon\}$.*
- (4) *The von Neumann algebra S_k is a type II factor. It is a finite factor if and only if the graph Γ is finite. Up to a scalar, the unique normal faithful semi-finite tracial weight of S_k is defined as $\text{Tr}(x) = \sum_{v \in V^\varepsilon} \langle \pi_k(x) p_v, p_v \rangle$ for any positive operator $x \in S_k$. Moreover, $\text{Tr}(y) = \tau_V \circ E(y)$ for any positive operator $y \in \text{Gr}_k \mathcal{P} \boxtimes \text{Gr}_k \mathcal{P}$.*

Proof. We assume that $k = 0$ and drop the subscript k . The general case can easily be deduced.

Proof of (1). Consider a vertex $v \in V^+$. It is obvious that $M(v)$ and $M(v)^{\text{op}}$ commute and generate the von Neumann algebra $T(v)$. Following [17, Corollary 4.12], we obtain that $M(v)$ is a II_1 factor. Let τ be the normal faithful tracial state of $M(v)$. Consider the map $E : \text{Gr} \mathcal{P} \boxtimes \text{Gr} \mathcal{P} \rightarrow \mathcal{P}_0^+$ defined in Section 2.1 and its restriction φ to $T(v) \cap \text{Gr} \mathcal{P} \boxtimes \text{Gr} \mathcal{P}$ that has values in $\mathbb{C} p_v \simeq \mathbb{C}$. Observe that $\varphi(a_1 b_1^{\text{op}} \cdots a_n b_n^{\text{op}}) = \tau(a_1 \cdots a_n) \tau(b_1 \cdots b_n)$ for any $a_1, \dots, a_n \in p_v \text{Gr} \mathcal{P}$ and $b_1^{\text{op}}, \dots, b_n^{\text{op}} \in p_v \text{Gr} \mathcal{P}^{\text{op}}$. This implies that $T(v)$ is isomorphic to $M(v) \bar{\otimes} M(v)^{\text{op}}$.

Proof of (2). Consider a loop l of length $2n + 2m$ that starts at an even vertex v . It defines a partial isometry $e_{a,b} \in \mathcal{P}_{n+m}^+ \subset B(C_{n+m}^+)$, where a is the path equal to the first half of the loop and b the opposite path equal to the second half of the loop. Consider the corresponding element $x_l := j(n, m)(e_{a,b}) \in S$. If $p_v x_l p_v = x_l$, then the $2n$ -th vertex in the loop l is also equal to v . This implies that l is the concatenation of two loops l_1 and l_2 where l_1 is the truncation of l for the $2n$ -th first edges and l_2 is the rest of the loop l . Let $y_i = e_{a_i, b_i}$ be the partial isometry of \mathcal{P}_n^+ if $i = 1$ and \mathcal{P}_m^+ if $i = 2$ such that the concatenation of the path a_i and the opposite of b_i is equal to the loop l_i for $i = 1, 2$. We have that $x_l = x_{l_1} \otimes x_{l_2}$, where $x_{l_1} = j(n, 0)(e_{a_1, b_1})$ and $x_{l_2} = j(0, m)(e_{a_2, b_2})$,

$$\text{i.e. } x_l = \frac{\text{---}2n\text{---}}{\text{---}2m\text{---}} \boxed{e_{a,b}} = \boxed{e_v} \frac{\text{---}2n\text{---}}{\text{---}2m\text{---}} \boxed{e_{a,b}} \boxed{e_v} = \frac{\text{---}2n\text{---}}{\text{---}2m\text{---}} \boxed{e_{a_1, b_1}} \boxed{e_{a_2, b_2}} = x_{l_1} \otimes x_{l_2}.$$

Therefore, $p_v D(n, m) p_v$ is contained in $T(v)$ for any $v \in V^+$ and $n, m \geq 0$. By density, we obtain that $p_v S p_v = T(v)$.

Proof of (3). Consider x in the relative commutant $T' \cap S$. For any $v, w \in V^+$ we have that $p_v x p_w = p_v p_w x$. Therefore, x is in T by (2). We obtain that x is in the center of T . We conclude by using (1).

Proof of (4). Consider an element x in the center of S . By (3) this element belongs to the center of T . Hence, there exists a bounded map $f : V^+ \rightarrow \mathbb{C}$ such that $x = \sum_{v \in V^+} f(v) p_v$. Consider two vertices v and w . There exists a path c of length n in Γ that starts at v and ends at w since the graph is connected. Consider the corresponding non-zero projection $e_{c,c} \in \mathcal{P}_n^+$ and its image $y := j(n, n)(e_{c,c})$ in S . Observe that $yx = f(w)y$ and $xy = f(v)y$. Therefore, f is constant and the center of S is trivial. We deduce that S is a factor. We have that $p_v S p_v$ is a II_1 factor by (1) and (2). Therefore, S is a type II factor. Consider the weight Tr of S defined by $Tr(x) = \sum_{v \in V^+} \langle \pi(x) p_v, p_v \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product of H . It is clear that this weight is normal, semi-finite, faithful, and sends p_v to $\mu_V(v)^2$ for any $v \in V^+$. Consider a positive operator $y \in Gr\mathcal{P} \boxtimes Gr\mathcal{P}$. Observe that

$$\begin{aligned} \sum_{v \in V^+} \langle \pi(y) p_v, p_v \rangle &= \sum_{v \in V^+} \tau_V \circ E(y p_v p_v^*) = \sum_{v \in V^+} \tau_V \circ E(y p_v) \\ &= \tau_V \circ E(y \sum_{v \in V^+} p_v) = \tau_V \circ E(y). \end{aligned}$$

Consider two loops l, k of length $2n + 2m$ and $2t + 2s$ in Γ that start at $v \in V^+$ and $w \in V^+$ respectively. Consider x_l and x_k the corresponding elements of S given by the inclusion maps $j(n, m)$ and $j(t, s)$. The identity (3) of Section 1 implies that $Tr(x_l x_k) = \delta_{n,t} \delta_{m,s} \mu_V(w) \mu_V(v) = Tr(x_k x_l)$, where $\delta_{n,t}$ is the Kronecker symbol. We obtain by density that Tr is tracial. By uniqueness of a normal faithful tracial weight on a type II factor, we have that S is a finite von Neumann algebra if and only if $Tr(1) < \infty$. Suppose that Γ is finite. Then $Tr(1) = \sum_{v \in V^+} Tr(p_v) = \sup_{v \in V^+} Tr(p_v) |V^+| < \infty$, where $|V^+|$ is the cardinal of V^+ . Hence, S is a II_1 factor. Suppose that Γ is infinite. Assume that $\{\mu_V(v) : v \in V^+\}$ is unbounded. Then, $Tr(1) \geq \sup_{v \in V^+} \mu_V(v)^2 = \infty$. Assume that $\{\mu_V(v) : v \in V^+\}$ is bounded by $C > 0$. Since $\delta^{-1} \leq \mu(a) \leq \delta$ and $\mu(a) = \mu_V(t(a))/\mu_V(s(a))$ for any $a \in C_1$, we obtain that $\{\mu_V(v) : v \in V^+\}$ is bounded below by a constant $D > 0$. Then, $Tr(1) = \sum_{v \in V^+} Tr(p_v) = \sum_{v \in V^+} \mu_V(v)^2 \geq D |V^+| = \infty$. Therefore, S is a II_∞ factor if Γ is infinite. \square

We denote by $L^2(S_k, Tr)$ the GNS Hilbert space associated to S_k and Tr and identify it with the Hilbert space H_k .

2.3. Planar algebras contained in a bipartite graph planar algebra. Consider a planar algebra \mathcal{Q} which embeds in the bipartite graph planar algebra \mathcal{P} . We identify \mathcal{Q} and its image in \mathcal{P} . Consider the $*$ -algebras $Gr_k \mathcal{Q}$, $Gr_k \mathcal{Q}^{\text{op}}$, and $Gr_k \mathcal{Q} \boxtimes Gr_k \mathcal{Q}$ that we identify with subalgebras of $Gr_k \mathcal{P}$, $Gr_k \mathcal{P}^{\text{op}}$, and $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ respectively. Denote by $S_k(\mathcal{Q})$, $M_k(\mathcal{Q})$, and $M_k(\mathcal{Q})^{\text{op}}$ the von Neumann subalgebras of S_k generated by $Gr_k \mathcal{Q} \boxtimes Gr_k \mathcal{Q}$, $Gr_k \mathcal{Q}$, and $Gr_k \mathcal{Q}^{\text{op}}$ respectively. Let $T_k(\mathcal{Q}) = M_k(\mathcal{Q}) \vee M_k(\mathcal{Q})^{\text{op}}$ be the von Neumann subalgebra of S_k generated by $M_k(\mathcal{Q})$ and $M_k(\mathcal{Q})^{\text{op}}$.

Proposition 2.3. *Let \mathcal{J} be the Temperley-Lieb-Jones planar algebra included in \mathcal{P} , i.e. the planar subalgebra of \mathcal{P} generated by tangles without inner discs. Then, $S_k(\mathcal{J}) \subset S_k$ is an irreducible subfactor. Moreover, $T_k(\mathcal{J})' \cap S_k = Z(T_k)$.*

Proof. We assume that $k = 0$ and drop the subscript k . The general case can easily be deduced. Consider the element $\cup \in D(1, 0)$ and $\cap \in D(0, 1)$. Denote by C and C^{op} the abelian von Neumann algebras generated by \cup and \cap respectively. Let $A = C \vee C^{\text{op}}$ be the abelian von Neumann algebra generated by C and C^{op} . Consider the von Neumann algebra $B = A \vee Z(T)$ generated by A and the set of projections $\{p_v : v \in V^+\}$. We claim that the relative commutant $S \cap A'$ is contained in B . Consider the following subspaces of S :

$$W^{t,n}(v) = \{x \in D(n, 0)p_v : \begin{array}{c} \text{---} \cup \text{---} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ \cap \text{---} \end{array} = 0\},$$

$$W_{b,m}(v) = \{x \in D(0, m)p_v : \begin{array}{c} \boxed{x} \\ \cap \text{---} \end{array} = \begin{array}{c} \text{---} \cup \text{---} \\ | \\ \boxed{x} \end{array} = 0\}, \text{ and}$$

$W_{b,m}^{t,n}(v, w)$ the space of $x \in p_w D(n, m)p_v$ such that

$$\begin{array}{c} \text{---} \cup \text{---} \\ | \\ \boxed{x} \end{array} = \begin{array}{c} \boxed{x} \\ \cap \text{---} \end{array} = \begin{array}{c} | \\ \boxed{x} \\ \cap \text{---} \end{array} = \begin{array}{c} | \\ \boxed{x} \\ \cap \text{---} \end{array} = 0, \text{ for any } n, m \geq 1, v, w \in V^+.$$

Observe that all those spaces are contained in $L^2(S, Tr)$ and are pairwise orthogonal. Denote by $W^t(v) = \bigoplus_{n \geq 1} W^{t,n}(v)$, $W_b(v) = \bigoplus_{m \geq 1} W_{b,m}(v)$, and $W_b^t(v, w) = \bigoplus_{n, m \geq 1} W_{b,m}^{t,n}(v, w)$ the direct sum of those spaces inside $L^2(S, Tr)$ for any $v, w \in V^+$. Let $H^t(v)$, $H_b(v)$, and $H_b^t(v, w)$ be the A -bimodules generated by $W^t(v)$, $W_b(v)$, and $W_b^t(v, w)$ respectively for any $v, w \in V^+$. Consider the A -bimodule generated by $D(0, 0)$ inside $L^2(S, Tr)$. It is isomorphic to $L^2(A) \otimes \ell^2(V^+)$ as a A -bimodule. We identify those two bimodules. A similar proof to [17, Theorem 4.9] shows that we have the following decomposition into A -bimodules:

$$(8) \quad L^2(S, Tr) = (L^2(A) \otimes \ell^2(V^+)) \oplus \bigoplus_{v \in V^+} H^t(v) \oplus \bigoplus_{v \in V^+} H_b(v) \oplus \bigoplus_{v, w \in V^+} H_b^t(v, w)$$

Consider $x \in S \cap A'$ and two different even vertices $v \neq w$. Observe that p_v and p_w commute with A . Furthermore, $p_v x p_w$ is in $L^2(S, Tr)$. Hence, $p_v x p_w$ is in $L^2(S, Tr) \cap A'$. The equality (8) implies that $p_v x p_w$ is in $H_b^t(v, w)$. Therefore, $p_v x p_w$ is a A -central vector of $H_b^t(v, w)$. Following [17, Theorem 4.9], we can prove that the A -bimodule $H_b^t(v, w)$ is isomorphic to $L^2(A) \otimes W_b^t(v, w) \otimes L^2(A)$. Therefore, it is isomorphic to a direct sum of the coarse A -bimodule $L^2(A) \otimes L^2(A)$. Thus, $H_b^t(v, w)$ does not admit any nonzero A -central vectors. Therefore, $p_v x p_w = 0$ for any $v \neq w$. The element $p_v x p_v$ is a A -central vector of $L^2(S, Tr)$. The equality (8) implies that $p_v x p_v \in L^2(A p_v) \oplus H^t(v) \oplus H_b(v) \oplus H_b^t(v, v)$. By the previous argument we have that the orthogonal component of $p_v x p_v$ inside $H_b^t(v, v)$ is equal to 0. The von Neumann algebra A is isomorphic to $C \bar{\otimes} C^{\text{op}}$. Let ξ be the orthogonal component of $p_v x p_v$ in $H^t(v)$. It is a C -central vector. We can prove that the A -bimodule $H^t(v)$ is isomorphic to $(L^2(C) \otimes W^t(v) \otimes L^2(C)) \otimes L^2(C^{\text{op}})$. In particular, the C -bimodule $H^t(v)$ is isomorphic to a direct sum of the coarse bimodule $L^2(C) \otimes L^2(C)$. Therefore, $\xi = 0$. A similar argument shows that the orthogonal component of $p_v x p_v$ in $H_b(v)$ is equal to 0. We obtain that $p_v x p_v$ is in the A -bimodule generated by p_v . Therefore, x belongs to B . This proves the claim.

Consider $x \in S \cap T(\mathcal{J})'$. Since A is contained in $T(\mathcal{J})$, we have that $x \in A' \cap S \subset B$. The element $p_v x$ is in $L^2(Ap_v)$ for any $v \in V^+$. It commutes with the two elements

$$\bigcap \text{ and } \bigcup.$$

A similar argument to [17, Corollary 4.11] shows that $p_v x \in \mathbb{C}p_v$ for any $v \in V^+$. Since $x = \sum_{v \in V^+} p_v x$, we obtain that $x \in Z(T)$. In particular, $T(\mathcal{J})' \cap S = Z(T)$.

Suppose that $x \in S \cap S(\mathcal{J})'$. By the argument of above, we have that $x \in Z(T)$. Hence, there exists a bounded function $f : V^+ \rightarrow \mathbb{C}$ such that $x = \sum_{v \in V^+} f(v)p_v$. Consider the element $\| \in D(1,1) \subset S(\mathcal{J})$ which commutes with x . Observe that $p_v \| p_w \neq 0$ if $v = w$ or if $d(v,w) = 2$, where d is the length metric of the graph Γ . Therefore, $p_v x \| p_w = f(v)p_v \| p_w = p_v \| x p_w = f(w)p_v \| p_w$ for any $v, w \in V^+$ such that $d(v,w) = 2$. This implies that $f(v) = f(w)$ if $d(v,w) = 2$. Since Γ is connected, we obtain that f is constant. Therefore, $x \in \mathbb{C}1$. \square

Part of the following proposition can be deduced from [8]. We provide a full proof for the convenience of the reader.

Proposition 2.4. *Consider a subfactor planar algebra \mathcal{Q} which is contained in \mathcal{P} . Then $S_k(\mathcal{Q})$ is a II_1 factor. If $v \in V^\epsilon$, then the map $tr : S_k(\mathcal{Q}) \rightarrow \mathbb{C}, x \mapsto \mu_V(v)^{-2} \langle \pi_k(x)p_v, p_v \rangle$ is the unique normal faithful tracial state of $S_k(\mathcal{Q})$.*

Proof. Let \mathcal{J} be the Temperley-Lieb-Jones planar algebra with modulus δ . We have a chain of inclusions $\mathcal{J} \subset \mathcal{Q} \subset \mathcal{P}$. This implies that we have the chain of inclusions $S_k(\mathcal{J}) \subset S_k(\mathcal{Q}) \subset S_k(\mathcal{P}) = S_k$. We obtain that $S_k(\mathcal{Q})$ is a factor since $S_k(\mathcal{J}) \subset S_k(\mathcal{P})$ is an irreducible subfactor by Proposition 2.3.

For any vertex $v \in V^\epsilon$ we consider the linear functional

$$tr_v : S_k(\mathcal{Q}) \rightarrow \mathbb{C}, x \mapsto \mu_V(v)^{-2} \langle \pi_k(x)p_v, p_v \rangle.$$

The linear function tr_v is a normal state since it is a vector state. Denote by B the $*$ -algebra $Gr_k \mathcal{Q} \boxtimes Gr_k \mathcal{Q}$.

Since \mathcal{Q} is a subfactor planar algebra, we have that \mathcal{Q}^ϵ is one dimensional and can be identified to the space of constant functions of $\mathcal{P}^\epsilon \simeq \ell^\infty(V^\epsilon)$. Consider $b \in B$ and the element $E(b) \in \mathcal{Q}^\epsilon$, where $E : Gr_k \mathcal{Q} \boxtimes Gr_k \mathcal{Q} \rightarrow \mathcal{P}_0^\epsilon$ is the map defined in Section 2.1. Observe that $tr_v(b)$ is the value of $E(b)$ at the vertex v . Since $E(b)$ is constant, we have that $tr_v(b) = tr_w(b)$ for any $v, w \in V^\epsilon$. By density, we obtain that $tr_v = tr_w$ for any $v, w \in V^\epsilon$.

We fix $v \in V^\epsilon$. Let us show that $tr := tr_v$ is tracial. By density, it is sufficient to show that $tr(ab) = tr(ba)$ for any $a, b \in B$. Consider $a, b \in B$ and denote by $a_{n,m}$ and $b_{n,m}$ their (n,m) -component in $D_k(n,m)$, $n, m \geq 0$. Recall that Tr is the weight of S_k defined in Proposition 2.2.4. Observe that

$$\begin{aligned} tr(ab) &= \sum_{n,m,i,j \geq 0} tr(a_{n,m} b_{i,j}) = \sum_{n,m,i,j \geq 0} \mu_V(v)^{-2} Tr(a_{n,m} b_{i,j} p_v) \\ &= \sum_{n,m \geq 0} \mu_V(v)^{-2} Tr(a_{n,m} b_{n,m} p_v) \text{ since } a_{n,m} \perp b_{i,j} p_v \text{ in } H_k \text{ if } (n,m) \neq (i,j) \\ &= \sum_{n,m \geq 0} tr(a_{n,m} b_{n,m}). \end{aligned}$$

Hence, it is sufficient to show that $tr(ab) = tr(ba)$ for $a, b \in D_k(n, m)$, $n, m \geq 0$. We fix $n, m \geq 0$, $a, b \in D_k(n, m)$, and $x, y \in \mathcal{Q}_{2k+n+m}^+$ such that $j_k(n, m)(x) = a$, $j_k(n, m)(y) = b$. Since \mathcal{Q} is spherical, we have that $tr(ab) = \tau_l(xy)$, where xy is the product of x and y in the planar algebra \mathcal{Q} and where τ_l is the left trace of \mathcal{Q} . By traciality of τ_l , we obtain that $\tau_l(xy) = \tau_l(yx) = tr(ba)$. Therefore, tr is a normal tracial state of the factor $S_k(\mathcal{Q})$. Since $S_k(\mathcal{Q})$ is infinite dimensional, we obtain that $S_k(\mathcal{Q})$ is a II_1 factor and its unique normal faithful tracial state is tr . \square

Proposition 2.5. *Consider a subfactor planar algebra \mathcal{Q} which is contained in \mathcal{P} . The tower of von Neumann algebras $M_0(\mathcal{Q}) \subset M_1(\mathcal{Q}) \subset M_2(\mathcal{Q}) \subset \dots$ is isomorphic to the tower constructed in [11]. The inclusion $T_k(\mathcal{Q}) \subset S_k(\mathcal{Q})$ is isomorphic to the k -th symmetric enveloping inclusion constructed in [8]. In particular, the subfactor planar algebra of $M_0(\mathcal{Q}) \subset M_1(\mathcal{Q})$ is isomorphic to \mathcal{Q} and the symmetric enveloping inclusion associated to $M_k(\mathcal{Q}) \subset M_{k+1}(\mathcal{Q})$ is isomorphic to $T_{k+1}(\mathcal{Q}) \subset S_{k+1}(\mathcal{Q})$.*

Proof. Let \mathcal{Q} be a subfactor planar algebra contained in \mathcal{P} . We put $B = Gr_k \mathcal{Q} \boxtimes Gr_k \mathcal{Q}$. In [8], to \mathcal{Q} is associated a tracial $*$ -algebra (V, \wedge, \dagger, Tr) where \wedge is the multiplication, \dagger the anti-linear involution, and Tr a tracial linear functional. They also consider another tracial $*$ -algebra (W, \star, \dagger, Tr') where W is a copy of V as a vector space. They provide a map $X : V \rightarrow W$ which is an isomorphism of tracial $*$ -algebras and consider a projection $p_{k,+} \in V$ that satisfies that $X(p_{k,+}) = p_{k,+}$. Observe that the vector spaces $p_{k,+}Wp_{k,+}$ and B are equal. Moreover, the $*$ -structures of $(p_{k,+}Wp_{k,+}, \star, \dagger)$ and (B, \cdot, \dagger) are defined by the same planar tangles. If we identify \mathcal{Q}_0^ϵ with the complex numbers inside \mathcal{P}_0^ϵ , then we have that our trace $tr : B \rightarrow \mathbb{C}$ considered in Proposition 2.4 is equal to the restriction to B of the map $E : Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P} \rightarrow \mathcal{P}_0^\epsilon$ defined in Section 2.1. Observe that by definition and the fact that \mathcal{Q} is spherical, we have that $\delta^{-2k} Tr'|_{p_{k,+}Wp_{k,+}}$ and $E|_B$ are the exact same maps. Therefore, the tracial $*$ -algebras $(p_{k,+}Wp_{k,+}, \star, \dagger, Tr')$ and (B, \cdot, \dagger, tr) are isomorphic. Thus (B, \cdot, \dagger, tr) is isomorphic to $(p_{k,+}Vp_{k,+}, \wedge, \dagger, Tr)$. It is proved in [8] that V acts by bounded operators on the GNS Hilbert space $L^2(V, Tr)$. This implies that the restriction of $\delta^{-2k} Tr$ to $p_{k,+}Vp_{k,+}$ that they denote by $\tau_k \boxtimes \tau_k$ is a faithful tracial state. Let L_k be the GNS completion of $p_{k,+}Vp_{k,+}$ with respect to $\tau_k \boxtimes \tau_k$ (which is denoted by $M_k \boxtimes M_k$ in [8]). It is proved that L_k is a II_1 factor with unique normal faithful tracial state $\tau_k \boxtimes \tau_k$.

Consider the von Neumann algebra $S_k(\mathcal{Q})$ which is the bicommutant of B inside S_k . The von Neumann algebra $S_k(\mathcal{Q})$ is a II_1 factor and its unique normal faithful tracial state is tr . Therefore, L_k and $S_k(\mathcal{Q})$ are both II_1 factors which contain a weakly dense $*$ -subalgebra $X^{-1}(B)$ and B respectively. Moreover, $\tau_k \boxtimes \tau_k \circ X^{-1}(b) = tr(b)$ for any $b \in B$. This implies that there exists an isomorphism $\Phi : L_k \rightarrow S_k(\mathcal{Q})$ such that $\Phi(X^{-1}(b)) = b$ for any $b \in B$. Let F_k be the von Neumann algebra generated by $Gr_k \mathcal{Q}$ and $Gr_k \mathcal{Q}^{\text{op}}$ inside L_k . It is easy to see that X^{-1} sends $Gr_k \mathcal{Q}$ and $Gr_k \mathcal{Q}^{\text{op}}$ to itself. This implies that $\Phi(F_k) = T_k(\mathcal{Q})$. Therefore, $T_k(\mathcal{Q}) \subset S_k(\mathcal{Q})$ is isomorphic to the k -th symmetric enveloping inclusion constructed in [8].

A similar proof shows that the tower of von Neumann algebras $M_0(\mathcal{Q}) \subset M_1(\mathcal{Q}) \subset M_2(\mathcal{Q}) \subset \dots$ is isomorphic to the tower constructed in [11]. The rest of the proposition follows from [11, Theorem 8] and [8, Theorem 3.3]. \square

Definition 2.6. If \mathcal{P} is a bipartite graph planar algebra or a subfactor planar algebra we say that $T_0 \subset S_0$ is the symmetric enveloping inclusion associated to \mathcal{P} and denote it by $T \subset S$.

3. FIXED POINT PLANAR ALGEBRAS

3.1. Actions of the automorphism group of the bipartite graph planar algebra.

Let (Γ, μ) be a weighted graph with modulus $\delta > 0$. Consider the bipartite graph planar algebra \mathcal{P} . An automorphism of \mathcal{P} is a sequence of maps $a = (a_n^\pm : n \geq 0)$ such that a_n^\pm is an automorphism of the von Neumann algebra \mathcal{P}_n^\pm and such that a commutes with the action of the planar tangles. We denote by $\text{Aut}(\mathcal{P})$ the automorphism group of \mathcal{P} .

Let $\text{Aut}(\Gamma)$ be the group of permutations of the vertices that conserve the number of edges connecting pairs of vertices and send even vertices to even vertices. We label each n -fold multiple edges by $\{1, \dots, n\}$ and extend each element of $\text{Aut}(\Gamma)$ by a permutation of the edges that preserves the labeling. Let $\text{Aut}(\Gamma, \mu)$ be the subgroup of $g \in \text{Aut}(\Gamma)$ such that $\mu(ga) = \mu(a)$ for any edge a . We extend the action of $\text{Aut}(\Gamma, \mu)$ to an action on all paths C_* of Γ that we denote as follows: $\text{Aut}(\Gamma, \mu) \times C_* \rightarrow C_*, (g, a) \mapsto ga$. Burstein proved that the following map $\gamma : \text{Aut}(\Gamma, \mu) \times \mathcal{P} \rightarrow \mathcal{P}, (g, e_{a,b}) \mapsto e_{ga,gb}$ defines an embedding of $\text{Aut}(\Gamma, \mu)$ into $\text{Aut}(\mathcal{P})$ [7].

Consider the von Neumann algebra \mathcal{P}_1^+ and its unitary group $L = U(\mathcal{P}_1^+)$. We recall an argument due to Burstein that explains how the group L embeds in $\text{Aut}(\mathcal{P})$. Consider the map $a \in C_* \mapsto \bar{a}$ that reverses the orientation of a path. It induces an injective *-morphisms $\text{Rev} : \mathcal{P}_1^\pm \rightarrow \mathcal{P}_1^\mp, e_{a,b} \mapsto e_{\bar{a},\bar{b}}$. We identify \mathcal{P}_n^\pm with its image in \mathcal{P}_{n+1}^+ and \mathcal{P}_n^- with its image in \mathcal{P}_{n+1}^+ . Consider the collection of shift operators $sh : \mathcal{P}_n^\pm \rightarrow \mathcal{P}_{n+2}^\pm, e_{a,b} \mapsto \sum_{c \in C_2^\pm} e_{ca,cb}, n \geq 0$. If $x \in \mathcal{P}_1^+$, we consider the element $x_1^+ = x \in \mathcal{P}_1^+, x_2^+ = x \text{Rev}(x) \in \mathcal{P}_2^+, x_{2n+1}^+ = x_{2n}^+ sh^n(x) \in \mathcal{P}_{2n+1}^+, x_{2n+2}^+ = x_{2n+1}^+ sh \circ \text{Rev}(x) \in \mathcal{P}_{2n+2}^+$ and $x_1^- = \text{Rev}(x) \in \mathcal{P}_1^-, x_2^- = x_1^- sh(x) \in \mathcal{P}_2^-, x_{2n+1}^- = x_{2n}^- sh^n(x_1^-) \in \mathcal{P}_{2n+1}^-, x_{2n+2}^- = x_{2n+1}^- sh^{n+1}(x) \in \mathcal{P}_{2n+2}^-$ for any $n \geq 1$. Consider $u \in L$, we have that u_n^\pm is a unitary of \mathcal{P}_n^\pm for any $n \geq 1$. We put $u_0^\pm = 1$ and consider the collection of automorphisms $(\text{Ad}(u_n^\pm) : n \geq 0)$, where $\text{Ad}(u_n^\pm)(a) = u_n^\pm a (u_n^\pm)^*$ for any $n \geq 0, a \in \mathcal{P}_n^\pm$. By [7, Section 3], the map $u \in L \mapsto (\text{Ad}(u_n^\pm) : n \geq 0)$ is an embedding of the group L into $\text{Aut}(\mathcal{P})$.

Consider the group $\text{Aut}(\Gamma, \mu)$ and its action γ on \mathcal{P} . This action γ defines an action of $\text{Aut}(\Gamma, \mu)$ on L that we continue to denote by γ . Consider the semi-direct product $L \rtimes \text{Aut}(\Gamma, \mu)$ with respect to this action. Burstein proved that those two subgroups generates $\text{Aut}(\mathcal{P})$ and that $\text{Aut}(\mathcal{P})$ is isomorphic to $L \rtimes \text{Aut}(\Gamma, \mu)$ [7]. We identify $\text{Aut}(\mathcal{P})$ and $L \rtimes \text{Aut}(\Gamma, \mu)$. If $g \in \text{Aut}(\mathcal{P}), x \in \mathcal{P}_n^\pm$, we denote by $g(x)$ the image of x under the automorphism g .

We consider the action of $\text{Aut}(\mathcal{P})$ on V given by

$$ug \cdot v = gv \text{ for any } u \in U(\mathcal{P}_1^+), g \in \text{Aut}(\Gamma, \mu), \text{ and } v \in V.$$

Here are some remarks regarding fixed point spaces contained in \mathcal{P} .

Remark 3.1. Consider a subgroup $G < \text{Aut}(\mathcal{P})$ and the fixed point space $\mathcal{Q} = \mathcal{P}^G$ under the action of G .

- The collection of fixed point spaces $(\mathcal{Q}_n^\pm = (\mathcal{P}_n^\pm)^G : n \geq 0)$ is a planar algebra.
- If G is contained in $\text{Aut}(\Gamma, \mu)$ and acts transitively on V^+ and V^- , then \mathcal{Q}_0^+ and \mathcal{Q}_0^- are one dimensional and \mathcal{Q} satisfies all the axioms of a subfactor planar algebra except the sphericity. In fact, \mathcal{Q} is a subfactor planar algebra if and only if for any $a \in C_1^\pm$ we have that $\mu(a)|\{\alpha \in G \cdot a : s(\alpha) = v^\pm\}| = \mu(\bar{a})|\{\alpha \in G \cdot \bar{a} : t(\alpha) = v^\mp\}|$ where $v^\pm \in V^\pm$ is a fixed pair of vertices. Indeed, the planar algebra \mathcal{Q} is spherical if and only if τ_r and τ_l coincide on \mathcal{Q}_1^\pm . Recall that $\{e_{a,b} : (a,b) \in ST_1^\pm\}$ is a system

of matrix units of \mathcal{P}_1^\pm , where $ST_1^\pm = \{(a, b) \in C_1^\pm \times C_1^\pm : s(a) = s(b) \text{ and } t(a) = t(b)\}$. Therefore, \mathcal{Q}_1^\pm is equal to the weak closure of $\text{Span}\{f_{a,b} = \sum_{(\alpha,\beta) \in G \cdot (a,b)} e_{\alpha,\beta} : (a, b) \in ST_1^\pm\}$. Consider $(a, b) \in ST_1^\pm$. We have that

$$\begin{aligned} \tau_l(f_{a,b}) &= \sum_{(\alpha,\beta) \in G \cdot (a,b)} \tau_l(e_{\alpha,\beta}) = \delta_{a,b} \sum_{\alpha \in G \cdot a} \mu(\bar{a}) e_{t(\alpha)} = \delta_{a,b} \sum_{w \in V^\mp} \sum_{\alpha \in G \cdot a : t(\alpha) = w} \mu(\bar{a}) e_w \\ &= \delta_{a,b} \mu(\bar{a}) |\{\alpha \in G \cdot a : t(\alpha) = v^\mp\}|. \end{aligned}$$

A similar computation shows that $\tau_r(f_{a,b}) = \delta_{a,b} \mu(a) |\{\alpha \in G \cdot a : s(\alpha) = v^\pm\}|$.

- If \mathcal{P}^G is a subfactor planar algebra, then G acts transitively on V^+ and V^- .
- In general $\mathcal{Q} = \mathcal{P}^G$ is reducible. We can deduce the following. Suppose there exists a subgroup $G < \text{Aut}(\mathcal{P})$ such that the fixed point space \mathcal{P}^G is an irreducible subfactor planar algebra. Then, the group $\text{Aut}(\Gamma, \mu)$ acts transitively on the set of positives edges C_1^+ . This implies that the weight μ is constant on C_1^+ .
- If we start with a countable locally finite undirected connected bipartite graph Γ that can have multiple edges between two vertices and a group $G < \text{Aut}(\Gamma)$ that acts transitively on V^+ and on V^- , then it was observed in [2, Proposition 2.5] that there exists a unique weight $\mu : C_1^+ \rightarrow \mathbb{R}_+^*$ such that \mathcal{P}^G is a subfactor planar algebra, where \mathcal{P} is the bipartite graph planar algebra associated to (Γ, μ) .

We fix a natural number $k \geq 0$.

Proposition 3.2. *Let $j_k(n, m) : \mathcal{P}_{n+m+2k}^+ \rightarrow S_k$ be the inclusion of the vector space \mathcal{P}_{n+m+2k}^+ in S_k for $n, m \geq 0$. There exists a group morphism $\sigma : \text{Aut}(\mathcal{P}) \rightarrow \text{Aut}(S_k)$ that satisfies that $\sigma_g \circ j_k(n, m)(x) = j_k(n, m)(g(x))$ for any $g \in \text{Aut}(\mathcal{P})$, $n, m \geq 0$, and $x \in \mathcal{P}_{n+m+2k}^+$. In particular, $\text{Aut}(\mathcal{P})$ acts on T_k , M_k and M_k^{op} . Moreover, the action of σ is minimal, i.e. the relative commutant $(S_k^{\text{Aut}(\mathcal{P})})' \cap S_k$ is trivial.*

Proof. For any $g \in \text{Aut}(\mathcal{P})$, $n, m \geq 0$, $x \in \mathcal{P}_{n+m+2k}^+$, we put $\sigma_g(j_k(n, m)(x)) = j_k(n, m)(g(x))$. Since any element $g \in \text{Aut}(\mathcal{P})$ commutes with the action of the planar operad, we obtain that σ_g is a $*$ -algebra morphism of $Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$. Observe that $p_v H_k$ is orthogonal to $p_w H_k$ for two different vertices $v, w \in V^+$. Since $\sum_{v \in V^+} p_v = 1$, we obtain that

$$(9) \quad H_k = \bigoplus_{v \in V^+} p_v H_k.$$

Consider $g \in \text{Aut}(\Gamma, \mu)$ and $x \in Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$ such that $\{v \in V : p_v x \neq 0\}$ is finite. Note that $\text{Tr}(xx^*) < \infty$. Observe that $\text{Tr} \circ \sigma_g = c_g \cdot \text{Tr}$ for any $g \in G$, where c_g is the ratio $\frac{\text{Tr}(p_{gw})}{\text{Tr}(p_w)} = \frac{\mu_V(gw)^2}{\mu_V(w)^2}$ which does not depend on $w \in V^+$. We put $U_g(x) = \sigma_g(x) / \sqrt{c_g}$. Observe that

$$c_g \|U_g(x)\|_2^2 = \|\sigma_g(x)\|_2^2 = \tau_V \circ E \circ \sigma_g(xx^*) = \tau_V \circ \sigma_g \circ E(xx^*),$$

since g commutes with the action of the planar operad. Since $\tau_V \circ \sigma_g = c_g \cdot \tau_V$, we obtain that $\|U_g(x)\|_2^2 = \tau_V \circ E(xx^*) = \|x\|_2^2$. This implies that U_g extends as an isometry of H_k . By definition, $U_{g^{-1}} \circ U_g(x) = x$ for any $n, m \geq 0$ and x in the range of $j_k(n, m)$. This implies that $U_{g^{-1}} \circ U_g$ is the identity operator. By symmetry, U_g is a unitary of H_k and $U_g^* = U_{g^{-1}}$. By definition and by a density argument we obtain that $U_g U_h = U_{gh}$ for any $g, h \in \text{Aut}(\Gamma, \mu)$. Consider the map $\text{Ad}(U_g)(x) = U_g x U_g^*$ for any $g \in \text{Aut}(\Gamma, \mu)$, $x \in B(H_k)$. Consider $x \in Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$, $g \in \text{Aut}(\Gamma, \mu)$, and $\xi \in H_k \cap Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$. We have that

$$\text{Ad}(U_g)(x)\xi = U_g x U_g^* \xi = U_g x \sqrt{c_g} \sigma_{g^{-1}}(\xi) = \sigma_g(x \sigma_{g^{-1}}(\xi)) = \sigma_g(x) \xi.$$

If $g \in L$, we define $U_g(x) = \sigma_g(x)$ for any $x \in H_k \cap Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}$. A similar proof shows that $Ad(U_g)(x) = \sigma_g(x)$ for any $x \in Gr_k \mathcal{P} \boxtimes Gr_k \mathcal{P}, g \in L$. One can check that $U_g U_l U_g^* = U_{\gamma_g(l)}$ for any $g \in \text{Aut}(\Gamma, \mu), l \in L$. This implies that $U : \text{Aut}(\mathcal{P}) \rightarrow U(H_k)$ is a unitary representation. By density, we obtain that $Ad(U_g)(S_k) = S_k$ for any $g \in \text{Aut}(\mathcal{P})$. We denote by σ_g the restriction to S_k of the automorphism $Ad(U_g), g \in \text{Aut}(\mathcal{P})$. We have that $\sigma_g \circ j_k(n, m)(x) = j_k(n, m)(g(x))$ for any $g \in \text{Aut}(\mathcal{P}), n, m \geq 0$, and $x \in \mathcal{P}_{n+m+2k}^+$. Hence, $\text{Aut}(\mathcal{P})$ acts on T_k, M_k and M_k^{op} by restricting $\sigma_g, g \in \text{Aut}(\mathcal{P})$.

If \mathcal{J} is the Temperley-Lieb-Jones planar algebra with modulus δ , then $S_k(\mathcal{J})$ is contained in the fixed point space $S_k^{\text{Aut}(\mathcal{P})}$. By Proposition 2.3, $S_k(\mathcal{J})$ is an irreducible subfactor of S_k . Therefore, $S_k^{\text{Aut}(\mathcal{P})} \subset S_k$ is an irreducible subfactor. \square

Proposition 3.3. *Consider a subgroup $G < \text{Aut}(\mathcal{P})$ and the fixed point planar algebra $\mathcal{Q} := \mathcal{P}^G$. Assume that \mathcal{Q} is a subfactor planar algebra. Consider the fixed point von Neumann algebras S_k^G, M_k^G , and $(M_k^{\text{op}})^G$. We have the equalities $S_k^G = S_k(\mathcal{Q}), M_k^G = M_k(\mathcal{Q})$, and $(M_k^{\text{op}})^G = M_k^{\text{op}}(\mathcal{Q})$, where $S_k(\mathcal{Q}), M_k(\mathcal{Q})$, and $M_k^{\text{op}}(\mathcal{Q})$ are the von Neumann subalgebras of S_k defined in Section 2.3. In particular, $M_0^G \subset M_1^G$ is a subfactor with subfactor planar algebra isomorphic to \mathcal{Q} . Furthermore, the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $M_0^G \vee (M_0^{\text{op}})^G \subset S_0^G$.*

Proof. We assume that $k = 0$ and drop the subscript k . The general case can easily be deduced. Let us show that $S^G = S(\mathcal{Q})$. By definition, $S(\mathcal{Q})$ is the weak closure of $B := Gr \mathcal{Q} \boxtimes Gr \mathcal{Q}$ inside S . Any element of B is G -invariant. Therefore, B is included in S^G and so does its weak closure $S(\mathcal{Q})$.

Consider a vertex $o \in V^+$ and the normal state $tr : S \rightarrow \mathbb{C}, x \mapsto Tr(p_o)^{-1} Tr(x p_o)$. By Proposition 2.4, $S(\mathcal{Q})$ is a II_1 factor and tr is its unique normal faithful tracial state. In particular, tr is faithful on $S(\mathcal{Q})$. Let us show that tr is faithful on S^G . Consider $x \in S^G$ such that $tr(xx^*) = 0$. The action of G on V^+ is transitive since \mathcal{P}^G is a subfactor planar algebra. Hence, for any $v \in V^+$, there exists $g_v \in G$ such that $g_v o = v$. Observe that

$$\begin{aligned} Tr(xx^*) &= \sum_{v \in V^+} Tr(xx^* p_v) = \sum_{v \in V^+} Tr(xx^* p_{g_v o}) \\ &= \sum_{v \in V^+} Tr(\sigma_{g_v}(xx^* p_o)) \text{ since } xx^* \text{ is } G\text{-invariant} \\ &= \sum_{v \in V^+} \frac{\mu_V(v)^2}{\mu_V(o)^2} Tr(xx^* p_o) = \sum_{v \in V^+} \mu_V(v)^2 tr(xx^*) = 0. \end{aligned}$$

Since Tr is faithful, we obtain that $x = 0$. Therefore, tr is a normal faithful state on S^G .

Let us show that $L^2(S(\mathcal{Q}), tr) = L^2(S^G, tr)$, where $L^2(S(\mathcal{Q}), tr)$ and $L^2(S^G, tr)$ are the GNS Hilbert space associated to $(S(\mathcal{Q}), tr)$ and (S^G, tr) . Consider ξ in the orthogonal complement of $L^2(S(\mathcal{Q}), tr)$ inside $L^2(S^G, tr)$. Observe that $Tr((p_o \xi)(p_o \xi)^*) = Tr(\xi \xi^* p_o) = Tr(p_o) tr(\xi \xi^*) < \infty$. Therefore, $p_o \xi \in L^2(S, Tr)$. The Hilbert space $L^2(S, Tr)$ is equal to the direct sum

$$\bigoplus_{n, m \geq 0} H(n, m).$$

Hence, there exists $x_m^n \in H(n, m), n, m \geq 0$ such that $p_o \xi = \sum_{n, m \geq 0} x_m^n$ where the sum converges in $L^2(S, Tr)$. Let us fixed $n, m \geq 0$. We have that $x_m^n \in p_o H(n, m)$. Observe

that

$$\{j(n, m)(e_{a,b}) : (a, b) \in ST_{n+m}^+, s(a) = o\}$$

is an orthogonal basis of $p_o H(n, m)$. Since Γ is locally finite, we obtain that this basis is finite. Therefore, $p_o H(n, m)$ is finite dimensional and is equal to $p_o D(n, m)$. Hence, there exists a unique $d \in e_o \mathcal{P}_{n+m}^+$ such that $j(n, m)(d) = x_m^n$. Recall that

$$\mathcal{P}_{n+m}^+ = \bigoplus_{v \in V^+, w \in V} B(\ell^2(C_{n+m}^+(v, w))), \text{ where } C_{n+m}^+(v, w) = \{a \in C_{n+m}^+ : s(a) = v, t(a) = w\}.$$

We have that $d \in \bigoplus_{w \in V} B(\ell^2(C_{n+m}^+(o, w)))$.

If $g \in \text{Aut}(\mathcal{P})$, then $g(d) \in \bigoplus_{w \in V} B(\ell^2(C_{n+m}^+(go, w)))$. Consider $G_o = \{g \in G : go = o\}$, and a system of representatives $\langle G/G_o \rangle$ of the quotient space G/G_o . The elements p_o and ξ are G_o -invariants. Therefore, $p_o \xi$ is G_o -invariant as an element of $L^2(S, Tr)$. Consider the orthogonal projection $P : L^2(S, Tr) \rightarrow H(n, m)$ and observe that P is G -equivariant. Consider $g \in G_o$. Since g fixes o , we have that $Tr \circ \sigma_g = Tr$ by uniqueness of the trace. Hence, σ_g and U_g coincide on $L^2(S, Tr) \cap S$, where U_g is the unitary defined in the proof of the previous proposition. Observe that $j(n, m)(g(d)) = \sigma_g \circ j(n, m)(d) = U_g(P(p_o \xi)) = P \circ U_g(p_o \xi) = P(p_o \xi) = j(n, m)(d)$. Therefore, d is G_o -invariant as an element of \mathcal{P}_{n+m}^+ .

We have that $go \neq ho$ for any $g \neq h$ in $\langle G/G_o \rangle$. This implies that the elements of $\{g(d) : g \in \langle G/G_o \rangle\}$ belongs to distinct summands of the von Neumann algebra

$$\mathcal{P}_{n+m}^+ = \bigoplus_{v \in V^+, w \in V} B(\ell^2(C_{n+m}^+(v, w))).$$

Moreover, $\|g(d)\| = \|d\|$ for any $g \in \text{Aut}(\mathcal{P})$ since g is an automorphism of the von Neumann algebra \mathcal{P}_{n+m}^+ . This implies that the sum $\sum_{g \in \langle G/G_o \rangle} g(d)$ converges for the strong operator topology to an element $\tilde{d} \in \mathcal{P}_{n+m}^+$. Let us show that \tilde{d} is G -invariant. Consider $f = \sum_{r \in R} r(d)$, where R is a system of representatives of G/G_o . For any $r \in R$ there exists $g_r \in \langle G/G_o \rangle, h_r \in G_o$ such that $r = g_r h_r$. Moreover, $\{g_r : r \in R\} = \langle G/G_o \rangle$. Observe that

$$f = \sum_{r \in R} r(d) = \sum_{r \in R} g_r h_r(d) = \sum_{r \in R} g_r(d) = \sum_{g \in \langle G/G_o \rangle} g(d) = \tilde{d}.$$

Therefore, \tilde{d} does not depend on the choice of the system of representatives $\langle G/G_o \rangle$. Consider $g \in G$. We have that $g(\tilde{d}) = \sum_{h \in \langle G/G_o \rangle} gh(d) = \sum_{r \in R} r(d)$, where $R = \{gh : h \in \langle G/G_o \rangle\}$. The set R is a system of representatives of G/G_o . Therefore, $g(\tilde{d}) = \tilde{d}$. Hence, \tilde{d} is a G -invariant element of \mathcal{P}_{n+m}^+ . This means that $\tilde{d} \in \mathcal{Q}_{n+m}^+$.

We put $y_m^n = j(n, m)(\tilde{d})$. By definition, $y_m^n \in B$. By assumption, $tr(\xi(y_m^n)^*) = 0$. Observe that

$$\begin{aligned} Tr(p_o)tr(\xi \xi^*) &= Tr(\xi \xi^* p_o) = Tr(\xi(p_o \xi)^*) = \sum_{k, l \geq 0} Tr(\xi(x_l^k)^*) \\ &= \sum_{k, l \geq 0} Tr(\xi(y_l^k)^* p_o) \text{ since } p_o y_l^k = x_l^k \\ &= Tr(p_o) \sum_{k, l \geq 0} tr(\xi(y_l^k)^*) = 0. \end{aligned}$$

We obtain that $L^2(S(\mathcal{Q}), tr) = L^2(S^G, tr)$ since the orthogonal complement of $L^2(S(\mathcal{Q}), tr)$ inside $L^2(S^G, tr)$ is trivial. Consider $\xi \in S^G$. Since tr is a faithful state, we have that $\xi \in L^2(S^G, tr)$. Hence, $\xi \in L^2(S(\mathcal{Q}), tr)$. Let us show that ξ is a bounded vector of the II_1 factor $S(\mathcal{Q})$. Consider the map $\theta : S(\mathcal{Q}) \rightarrow L^2(S(\mathcal{Q}), tr), y \mapsto y\xi$. Observe that

$$\begin{aligned} tr(\theta(y)\theta(y)^*) &= Tr(p_o)^{-1}Tr(y\xi\xi^*y^*p_o) \leq Tr(p_o)^{-1}\|\xi\xi^*\|_S Tr(yy^*p_o) \\ &\leq \|\xi\|_S^2 tr(yy^*) \text{ for any } y \in S(\mathcal{Q}). \end{aligned}$$

Therefore, θ extends as a bounded linear operator from $L^2(S(\mathcal{Q}), tr)$ to $L^2(S(\mathcal{Q}), tr)$. Since $S(\mathcal{Q})$ is a II_1 factor and tr is its unique normal faithful tracial state, we obtain that $\xi \in S(\mathcal{Q})$ by [18, Theorem 1.2.4.2]. Therefore, $S(\mathcal{Q}) = S^G$.

A similar proof show that $M(\mathcal{Q}) = M^G$ and $M(\mathcal{Q})^{\text{op}} = (M^{\text{op}})^G$. By [11] and Proposition 2.5, \mathcal{Q} is isomorphic to the subfactor planar algebra of $M_0(\mathcal{Q}) \subset M_1(\mathcal{Q})$. Therefore, it is isomorphic to the subfactor planar algebra of $M_0^G \subset M_1^G$. The symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $M_0(\mathcal{Q}) \vee M_0(\mathcal{Q})^{\text{op}} \subset S_0(\mathcal{Q})$ by Proposition 2.5. Therefore, it is isomorphic to $M_0^G \vee (M_0^{\text{op}})^G \subset S_0^G$. \square

3.2. Proof of Theorem A. Let (Γ, μ) be a weighted graph with modulus δ and let $G < \text{Aut}(\mathcal{P})$ be a countable or closed subgroup. Let \mathcal{P} be the bipartite graph planar algebra associated to (Γ, μ) and let $T \subset S$ be its symmetric enveloping inclusion. Consider the fixed point space $\mathcal{Q} = \mathcal{P}^G$ that we assume to be a subfactor planar algebra. This implies that the action of G on V^+ is necessarily transitive and that $T \subset S$ is an irreducible subfactor. If $K < L$ is an inclusion of groups, we denote by $\langle L/K \rangle$ (resp. $\langle K \setminus L/K \rangle$) a system of representatives of the coset space L/K (resp. $K \setminus L/K$). If $v, w \in V^+$, then G_v (resp. $G_{v,w}$) denotes the subgroup of G that fixes the vertex v (resp. fixes the vertices v and w). Let $o \in V^+$ be a fixed vertex. Note that G/G_o is in bijection with V^+ since G acts transitively on V^+ . Hence, $\{p_{go} : g \in \langle G/G_o \rangle\}$ is the set of minimal central projections of T .

Remark 3.4. • The subgroup $G_o < G$ is almost normal. Indeed, consider $g \in G$. We need to show that $G_o \cap gG_o g^{-1}$ is a finite index subgroup of G_o and $gG_o g^{-1}$. Observe that $G_o \cap gG_o g^{-1} = G_{o,go} = \{h \in G : hgo = go \text{ and } ho = o\}$. Consider the map $h \in G_o \mapsto hgo \in V$. We have that $G_o/G_{o,go}$ is in bijection with the orbit $G_o \cdot go$. But $G_o \cdot go$ is contained in the sphere centered in o with radius equal to the distance between o and go . This sphere is a finite set since Γ is locally finite. Therefore, $G_{o,go}$ is a finite index subgroup of G_o . A similar argument shows that $G_{o,go}$ is a finite index subgroup of $gG_o g^{-1}$. Therefore, $G_o < G$ is an almost normal subgroup.

- By Propositions 2.4 and 3.3, the unique normal faithful tracial state of the II_1 factor S^G is $tr : S^G \rightarrow \mathbb{C}, x \mapsto Tr(p_v)^{-1}Tr(p_v x)$ where $v \in V^+$.

The next proposition gives a decomposition of the T^G -bimodule $L^2(S^G, tr)$.

Proposition 3.5. *We define the Hilbert space H_g which is the closure in $L^2(S, Tr)$ of $p_o S^G p_{go}$ for any $g \in G$. The following assertions are true.*

- (1) *The Hilbert space H_g is a bifinite T^G -bimodule for any $g \in G$.*
- (2) *The T^G -bimodules $L^2(S^G, tr)$ and $\bigoplus_{g \in \langle G_o \setminus G/G_o \rangle} H_g$ are isomorphic.*

Proof. Proof of (1). Consider $g \in G$. Since $Tr(p_o) < \infty$, we have that $p_o S$ is contained in $L^2(S, Tr)$. Therefore, H_g is well defined. Since p_{go} and p_o commute with T^G ,

we have that H_g is a T^G -bimodule. Suppose that $Tr(p_o) \leq Tr(p_{go})$. Since $S^{G_{o,go}}$ is a factor by Proposition 3.2, there exists some elements $m_1, \dots, m_n \in S^{G_{o,go}}$ such that $m_1 m_1^* = p_o, m_i m_i^* \leq p_o, 2 \leq i \leq n$ and $\sum_{j=1}^n m_j^* m_j = p_{go}$. Since $G_o < G$ is an almost normal subgroup by Remark 3.4, we have that the index $[G_o : G_{o,go}]$ is finite. Therefore, $[T(o)^{G_{o,go}} : T(o)^{G_o}] < \infty$. Hence, there exists a finite subset $F \subset T(o)^{G_{o,go}}$ such that $T(o)^{G_{o,go}} = \text{Span}(T(o)^{G_o} \cdot F)$. Similarly, there exists a finite subset $E \subset T(go)^{G_{o,go}}$ such that $T(go)^{G_{o,go}} = \text{Span}(E \cdot T(go)^{G_{go}})$. Observe that

$$p_{go} T^G = T^G p_{go} = T(go)^{G_{go}}.$$

Consider $x \in p_o S^G p_{go}$. We have that $x = m_1(m_1^* x)$ and $m_1^* x \in p_{go} S p_{go} = T(go)$. Moreover, $m_1^* x$ is $G_{o,go}$ -invariant. Therefore, $m_1^* x$ is in $T(go)^{G_{o,go}} = \text{Span}(E \cdot T(go)^{G_{go}})$. Hence, $x \in \text{Span}(m_1 \cdot E \cdot T(go)^{G_{go}}) = \text{Span}(m_1 \cdot E \cdot T^G)$.

Consider $1 \leq i \leq n$. Observe that $x = \sum_{j=1}^n x m_j^* m_j$ and $x m_i^* \in p_o S p_o = T(o)$. Moreover, $x m_i^*$ is $G_{o,go}$ -invariant. Hence, $x m_i^*$ is in $T(o)^{G_{o,go}} = \text{Span}(T(o)^{G_o} \cdot F) = \text{Span}(T^G \cdot F)$. Therefore, x is in $\text{Span}(T^G \cdot F \cdot \{m_j : 1 \leq j \leq n\})$. Therefore, H_g is a bifinite T^G -bimodule.

Proof of (2). Consider the map $\beta : L^2(S^G, tr) \longrightarrow L^2(S, Tr)$ such that

$$\beta(x) = \sum_{g \in \langle G_o \setminus G / G_o \rangle} p_o x p_{go} \sqrt{\frac{[G_o : G_{o,go}]}{Tr(p_o)}}$$

for any $x \in S^G$, where the sum converges for the L^2 -norm. Let us show that β is an isometry. Let $\sigma : G \longrightarrow \text{Aut}(S)$ be the action of G on S . Observe that $Tr \circ \sigma_g = c_g \cdot Tr$ for any $g \in G$, where c_g is the ratio $\frac{Tr(p_{gw})}{Tr(p_w)}$ which does not depend on $w \in V^+$. In particular,

$$(10) \quad Tr \circ \sigma_g = Tr \text{ if } g \in G \text{ fixes a vertex of } V^+.$$

Consider $x \in S^G$. We have that

$$\begin{aligned} Tr(\beta(x)\beta(x)^*) &= \sum_{g,h \in \langle G_o \setminus G / G_o \rangle} \frac{\sqrt{[G_o : G_{o,go}][G_o : G_{o,ho}]}}{Tr(p_o)} Tr(p_o x p_{go} p_{ho} x^* p_o) \\ &= \sum_{g \in \langle G_o \setminus G / G_o \rangle} \frac{[G_o : G_{o,go}]}{Tr(p_o)} Tr(p_o x p_{go} x^*). \end{aligned}$$

Consider $g, h \in G$ such that $G_o g G_o = G_o h G_o$. Then, there exists $k \in G_o$ such that $g G_o = k h G_o$. Observe that

$$\begin{aligned} Tr(p_o x p_{go} x^*) &= Tr(p_o x p_{k h o} x^*) = Tr \circ \sigma_k(p_o x p_{h o} x^*) \\ &= Tr(p_o x p_{h o} x^*) \text{ by (10).} \end{aligned}$$

Therefore, $Tr(p_o x p_{g_o} x^*) = [G_o : G_{o,g_o}]^{-1} \sum_{k \in \langle G_o / G_{o,g_o} \rangle} Tr(p_o x p_{k g_o} x^*)$. Hence,

$$\begin{aligned} Tr(\beta(x)\beta(x)^*) &= \sum_{g \in \langle G_o \setminus G / G_o \rangle} \sum_{k \in \langle G_o / G_{o,g_o} \rangle} \frac{Tr(p_o x p_{k g_o} x^*)}{Tr(p_o)} \\ &= \sum_{s \in \langle G / G_o \rangle} \frac{Tr(p_o x p_{s o} x^*)}{Tr(p_o)} \text{ for a system of representatives } \langle G / G_o \rangle \\ &= \frac{Tr(p_o x x^*)}{Tr(p_o)} = tr(x x^*). \end{aligned}$$

Therefore, β defines an isometry from $L^2(S^G, tr)$ to $L^2(S, Tr)$.

The map β is T^G -bimodular because p_o and p_{g_o} commute with T^G for any $g \in G$. Note that H_g is orthogonal to H_k if $g G_o \neq k G_o$. Therefore, the T^G -bimodules $\{H_l : l \in \langle G_o \setminus G / G_o \rangle\}$ are pairwise orthogonal. We have that $Im \beta \subset \bigoplus_{l \in \langle G_o \setminus G / G_o \rangle} H_l$ by definition. Let us show that the range of β is equal to $\bigoplus_{l \in \langle G_o \setminus G / G_o \rangle} H_l$.

We fix $g \in \langle G_o \setminus G / G_o \rangle$. Consider $y \in p_o S^G p_{g_o} = H_g \cap S$. Let us show that the sum $\sum_{s \in \langle G / G_{o,g_o} \rangle} \sigma_s(y)$ converges for the weak operator topology of S . Consider $\varepsilon > 0, \xi, \eta \in L^2(S, Tr)$ such that $\|\xi\|_2 = \|\eta\|_2 = 1$. If $A \subset V^+$, we denote by A^c its complement and by p_A the projection $\sum_{v \in A} p_v$. Since $1 = \|\xi\|_2^2 = \sum_{v \in V^+} \|p_v \xi\|_2^2 = \sum_{v \in V^+} \|p_v \eta\|_2^2$, there exists a finite subset $A \subset V^+$ such that $\|p_{A^c} \xi\|_2^2, \|p_{A^c} \eta\|_2^2 < \varepsilon$. Consider the set $E = \{s \in \langle G / G_{o,g_o} \rangle : s o \in A \text{ or } s g o \in A\}$. Note that its cardinal $|E|$ is smaller than $|A|([G_o : G_{o,g_o}] + [G_{g_o} : G_{o,g_o}]) < \infty$. Denote by E^c the complement of E inside $\langle G / G_{o,g_o} \rangle$. Observe that

$$\begin{aligned} |\langle \sum_{s \in E^c} \sigma_s(y) \xi, \eta \rangle| &\leq \sum_{s \in E^c} |\langle \sigma_s(y) \xi, \eta \rangle| \leq \sum_{s \in E^c} |\langle \sigma_s(y) p_{s g o} \xi, p_{s o} \eta \rangle| \text{ since } y \in p_o S p_{g_o} \\ &\leq \sum_{s \in E^c} \|\sigma_s(y) p_{s g o} \xi\|_2 \|p_{s o} \eta\|_2 \text{ by the Cauchy-Schwarz inequality} \\ &\leq \sum_{s \in E^c} \|y\| \|p_{s g o} \xi\|_2 \|p_{s o} \eta\|_2 \\ &\leq \|y\| \sqrt{\sum_{s \in E^c} \|p_{s g o} \xi\|_2^2} \sqrt{\sum_{t \in E^c} \|p_{t o} \eta\|_2^2} \text{ by the Cauchy-Schwarz inequality} \\ &\leq \|y\| \sqrt{\sum_{w \in A^c} |\{s \in E^c : s g o = w\}| \|p_w \xi\|_2^2} \sqrt{\sum_{w \in A^c} |\{t \in E^c : t o = w\}| \|p_w \eta\|_2^2} \end{aligned}$$

Consider $w \in V^+, r \in G$ such that $rw = o$ and $F = \{t \in E^c : t o = w\}$. Observe that $r \cdot F \subset G_o \cap r \langle G / G_{o,g_o} \rangle$. Therefore, $|F| = |r \cdot F| \leq |G_o \cap r \langle G / G_{o,g_o} \rangle| = [G_o : G_{o,g_o}]$, since $|G_o \cap R| = [G_o : G_{o,g_o}]$ for any system of representatives R of $G / G_{o,g_o}$. Similarly, $|\{s \in E^c : s g o = w\}| \leq [G_{g_o} : G_{o,g_o}]$. Therefore,

$$\begin{aligned} |\langle \sum_{s \in E^c} \sigma_s(y) \xi, \eta \rangle| &\leq \sqrt{[G_o : G_{o,g_o}][G_{g_o} : G_{o,g_o}]} \|p_{A^c} \xi\|_2 \|p_{A^c} \eta\|_2 \\ &\leq \varepsilon \sqrt{[G_o : G_{o,g_o}][G_{g_o} : G_{o,g_o}]}. \end{aligned}$$

This implies that for any $\delta > 0$ there exists a finite set $E \subset \langle G/G_{o,go} \rangle$ such that for any sets I, J that contains E we have that $|\langle \sum_{s \in I} \sigma_s(y) \xi, \eta \rangle - \langle \sum_{t \in J} \sigma_t(y) \xi, \eta \rangle| < \delta$. Hence, $((\sum_{s \in I} \sigma_s(y) : I \subset \langle G/G_{o,go} \rangle))$ is a Cauchy filter in S with respect to the weak uniform structure. This implies that the sum $\sum_{s \in \langle G/G_{o,go} \rangle} \sigma_s(y)$ converges weakly in S , since S is complete for the weak uniform structure. We denote by $\Theta_g(y)$ this limit. Let us show that $\Theta_g(y)$ is G -invariant. Suppose that R is a system of representatives of $G/G_{o,go}$. For any $r \in R$ there exists a unique $a_r \in \langle G/G_{o,go} \rangle$ and $b_r \in G_{o,go}$ such that $r = a_r b_r$. We have that $\sigma_r(y) = \sigma_{a_r b_r}(y) = \sigma_{a_r} \circ \sigma_{b_r}(y) = \sigma_{a_r}(y)$ since y is $G_{o,go}$ -invariant. Moreover, $\{a_r : r \in R\} = \langle G/G_{o,go} \rangle$. Therefore, $\Theta_g(y) = \sum_{r \in R} \sigma_r(y)$. Hence, Θ_g does not depend on the choice of the system of representatives of $G/G_{o,go}$. Consider $a \in G$. We have that $\sigma_a \circ \Theta_g(y) = \sum_{r \in R} \sigma_{ar}(y)$, where $R = a \cdot \langle G/G_{o,go} \rangle$. Since R is a system of representatives, we obtain that $\sigma_a \circ \Theta_g(y) = \Theta_g(y)$. Hence, $\Theta_g(y)$ is G -invariant.

Let $\langle G_o/G_{o,go} \rangle$ be a system of representatives of $G_o/G_{o,go}$. Observe that

$$\begin{aligned}
\text{tr}(\Theta_g(y)\Theta_g(y)^*) &= \text{Tr}(p_o)^{-1} \text{Tr}(\Theta_g(y)\Theta_g(y)^* p_o) \\
&= \text{Tr}(p_o)^{-1} \sum_{s,t \in \langle G/G_{o,go} \rangle} \text{Tr}(\sigma_s(y)\sigma_t(y)^* p_o) \\
&= \text{Tr}(p_o)^{-1} \sum_{s \in G_o \cap \langle G/G_{o,go} \rangle} \text{Tr}(\sigma_s(y)\sigma_s(y)^*) \text{ since } y \in p_o S p_{go} \\
&= \text{Tr}(p_o)^{-1} \sum_{s \in G_o \cap \langle G/G_{o,go} \rangle} \text{Tr}(yy^*) \text{ by (10)} \\
&= \frac{[G_o : G_{o,go}]}{\text{Tr}(p_o)} \|y\|_2^2.
\end{aligned}$$

Hence, Θ_g extends to an injective bounded linear operator from H_g to $L^2(S^G, \text{tr})$. Observe that Θ_g is a T^G -bimodular map. Consider $z \in S^G$ and $y = p_o z p_{go}$. We have that

$$\begin{aligned}
\beta \circ \Theta_g(y) &= \sum_{t \in \langle G_o \setminus G/G_o \rangle} \sum_{s \in \langle G/G_{o,go} \rangle} p_o \sigma_s(y) p_{to} \sqrt{\frac{[G_o : G_{o,to}]}{\text{Tr}(p_o)}} \\
&= \sum_{t \in \langle G_o \setminus G/G_o \rangle} \sum_{s \in \langle G/G_{o,go} \rangle} \delta_{o,so} \delta_{sgo,to} p_o z p_{to} \sqrt{\frac{[G_o : G_{o,to}]}{\text{Tr}(p_o)}},
\end{aligned}$$

where $\delta_{a,b}$ is the Kronecker symbol. Observe that if $o = so$ and $sgo = to$, then $s \in G_o$ and $G_o g G_o = G_o t G_o$. Therefore,

$$\begin{aligned}
\beta \circ \Theta_g(y) &= \sum_{s \in G_o \cap \langle G/G_{o,go} \rangle} \delta_{sgo,go} p_o z p_{go} \sqrt{\frac{[G_o : G_{o,go}]}{\text{Tr}(p_o)}} \\
&= y \sqrt{\frac{[G_o : G_{o,go}]}{\text{Tr}(p_o)}}
\end{aligned}$$

We obtain that $\beta \circ \Theta_g = \sqrt{\frac{[G_o : G_{o,go}]}{\text{Tr}(p_o)}} \text{Id}_{H_g}$ by a density argument. Therefore, H_g is contained in $\text{Im} \beta$ for any $g \in \langle G_o \setminus G/G_o \rangle$. Since $\text{Im} \beta \subset \bigoplus_{l \in \langle G_o \setminus G/G_o \rangle} H_l$ and β is an isometry, we

obtain that $Im\beta = \bigoplus_{l \in \langle G_o \backslash G/G_o \rangle} H_l$. Hence, β realizes an isomorphism of T^G -bimodules from $L^2(S^G, tr)$ onto $\bigoplus_{l \in \langle G_o \backslash G/G_o \rangle} H_l$. \square

Proposition 3.6. *Let $f : G \rightarrow \mathbb{C}$ be a unital bounded G_o -bi-invariant map. Denote by $\bar{f} : G/G_o \rightarrow \mathbb{C}$ the induced map on the coset space. The following assertions are true.*

- (1) *There exists a unique normal trace-preserving unital T -bimodular map $\phi_f : S \rightarrow S$ such that $\phi_f(x) = f(h^{-1}g)x$ for any $x \in p_{go}Sp_{ho}$ and $g, h \in G$.*
- (2) *If f is positive definite, then ϕ_f is completely positive.*
- (3) *The restriction of ϕ_f to S^G defines a trace-preserving T^G -bimodular map $\psi_f : S^G \rightarrow S^G$.*
- (4) *The map \bar{f} has finite support if and only if $\psi_f(S^G)$ is a bifinite T^G -bimodule.*

Proof. Proof of (1). The map ϕ_f is well defined on the weakly dense $*$ -subalgebra $B = \{x \in Gr\mathcal{P} \boxtimes Gr\mathcal{P} : \exists F \subset \langle G/G_o \rangle \text{ finite such that } x = \sum_{s,t \in F} p_{so}xp_{to}\}$. We have that

$$\begin{aligned} Tr \circ \phi_f(x) &= Tr\left(\sum_{g,h \in F} f(h^{-1}g)p_{go}xp_{ho}\right) = \sum_{g \in F} Tr(f(g^{-1}g)p_{go}xp_{go}) \\ &= \sum_{g \in F} Tr(p_{go}xp_{go}) = Tr(x), \text{ for any } x \in B. \end{aligned}$$

Therefore, ϕ_f is trace-preserving on B . The map ϕ_f is ultraweakly continuous and extends to a normal trace-preserving map $\phi_f : S \rightarrow S$. This map is clearly unital since f is unital. Consider $t_1, t_2 \in T$ and $x \in \sum_{g,h \in F} p_{go}Sp_{ho}$ where F is a finite set. Observe that $t_1p_{go}xp_{ho}t_2 \in p_{go}Sp_{ho}$ for any $g, h \in \langle G/G_o \rangle$. Therefore,

$$\phi_f(t_1xt_2) = \phi_f\left(\sum_{g,h \in F} t_1p_{go}xp_{ho}t_2\right) = \sum_{g,h \in F} f(h^{-1}g)t_1p_{go}xp_{ho}t_2 = t_1\phi_f(x)t_2.$$

Hence, by a density argument we have that the map ϕ_f is T -bimodular.

Proof of (2). Suppose that f is positive definite. There exists a Hilbert space \mathcal{H} and a continuous right- G_o -invariant bounded map $\xi : G \rightarrow \mathcal{H}$ such that $f(h^{-1}g) = \langle \xi(h), \xi(g) \rangle$ for any $g, h \in G$. Consider the $*$ -representation $\rho : S \rightarrow B(L^2(S, Tr) \otimes \mathcal{H})$ defined as $\rho(x)(a \otimes \zeta) = xa \otimes \zeta$ for any $x \in S$, $a \in L^2(S, Tr)$ and $\zeta \in \mathcal{H}$. Consider the bounded operator $A : L^2(S, Tr) \rightarrow L^2(S, Tr) \otimes \mathcal{H}$, $a \in p_{go}L^2(S, Tr) \mapsto a \otimes \xi(g)$, for any $g \in \langle G/G_o \rangle$. One can check that $\phi_f(x) = A^*\rho(x)A$ for any $x \in S$. Therefore, ϕ_f is completely positive.

Proof of (3). Consider the action $\sigma : G \rightarrow \text{Aut}(S)$. Since ϕ_f does not depend on the choice of system of representatives $\langle G/G_o \rangle$ we get that $\phi_f \circ \sigma_g = \sigma_g \circ \phi_f$ for any $g \in G$. This implies that $\phi_f(S^G)$ is contained inside S^G . The map ψ_f is T^G -bimodular since it is the restriction of a T -bimodular map. Consider the normal map $E_T^S : S \rightarrow S, x \mapsto \sum_{g \in \langle G/G_o \rangle} p_{go}xp_{go}$. The unique normal faithful tracial state tr of S^G is the restriction of the map $\omega : S \rightarrow \mathbb{C}, x \mapsto Tr(p_o)^{-1}Tr(xp_o)$. It is easy to see that $\omega \circ E_T^S = \omega$ and $E_T^S \circ \phi_f = E_T^S$. Therefore,

$$tr \circ \psi_f(x) = \omega \circ \phi_f(x) = \omega \circ E_T^S \circ \phi_f(x) = \omega(x) = tr(x),$$

for any $x \in S^G$.

Proof of (4). Since G/G_o is an almost normal subgroup, we have that \bar{f} is finitely supported if and only if the support $\text{supp}(f)$ is contained in finitely many double cosets of $G_o \backslash G/G_o$. Since f is G_o -bi-invariant, we have that $\text{supp}(f)$ is stable by left and right

multiplication by G_o . Hence, there exists $E \subset \langle G_o \backslash G / G_o \rangle$ such that $\text{supp}(f) = G_o \cdot E \cdot G_o$. Observe that the norm closure of the image of ψ_f in $L^2(S^G, tr)$ is isomorphic to $\bigoplus_{g \in E} H_g$. Proposition 3.5 implies the result. \square

Proof of Theorem A. Consider $(\Gamma, \mu), G, G_o, \mathcal{P}$ as above such that G is amenable. Denote by $\mathcal{Q} = \mathcal{P}^G$ the subfactor planar algebra equal to the fixed point space under the action of G . As observed in Section 1.3, the subgroup $G_o < G$ is co-amenable since G is amenable. By Section 1.3, there exists a sequence of positive definite G_o -bi-invariant maps $f_n : G \rightarrow \mathbb{C}, n \geq 0$ with support contained in a finite union of right G_o -cosets and that converges pointwise to 1. By Proposition 3.6, the collection $(\psi_n := \psi_{f_n}, n \geq 0)$ is a sequence of completely positive T^G -bimodular maps from S^G to S^G such that $\psi_n(S^G)$ is a bifinite T^G -bimodule for any $n \geq 0$. It is easy to see that $\lim_{n \rightarrow \infty} \|\psi_n(x) - x\|_2 = 0$ for any $x \in S^G$. Therefore, $T^G \subset S^G$ is co-amenable.

Consider the inclusion $M^G \vee (M^{\text{op}})^G \subset T^G$. The map $\Theta : T(o)^{G_o} \rightarrow S, x \mapsto \sum_{s \in \langle G/G_o \rangle} \sigma_s(x)$ realizes an isomorphism from $T(o)^{G_o}$ onto T^G such that $\Theta(M(o)^{G_o}) = M^G$ and $\Theta((M(o)^{\text{op}})^{G_o}) = (M^{\text{op}})^G$. Therefore, the inclusion $M^G \vee (M^{\text{op}})^G \subset T^G$ is isomorphic to $T(o)^{G_o \times G_o} \subset T(o)^{G_o}$ for the action $\sigma \otimes \sigma^{\text{op}} : G_o \times G_o \rightarrow \text{Aut}(M(o)) \times \text{Aut}(M(o)^{\text{op}}) < \text{Aut}(T(o))$. The group G_o is amenable since it is a closed subgroup of the amenable group G . By the same observation made in Section 1.3, we have that the subgroup $G_o \subset G_o \times G_o$ given by the diagonal inclusion is co-amenable since $G_o \times G_o$ is amenable. Therefore, the inclusion $M^G \vee (M^{\text{op}})^G \subset T^G$ is co-amenable by [20, Proposition 6]. The composition of two co-amenable inclusions is co-amenable by [25, Theorem 3.2.4.1]. Therefore, $M^G \vee (M^{\text{op}})^G \subset S^G$ is co-amenable. Proposition 3.3 implies that this later inclusion is isomorphic to the symmetric enveloping inclusion of \mathcal{Q} . Therefore, the subfactor planar algebra \mathcal{Q} is amenable. \square

We end this section with an observation regarding the principal graph of \mathcal{Q} .

Proposition 3.7. *Let $(\Gamma, \mu), G, G_o$, and \mathcal{Q} satisfying the assumptions of this section. Moreover, assume that G is contained in $\text{Aut}(\Gamma, \mu)$. Then the subfactor planar algebra \mathcal{Q} has finite depth if and only if the graph Γ is finite.*

Proof. Recall that a subfactor planar algebra \mathcal{Q} has finite depth if and only if its symmetric enveloping inclusion is a finite index subfactor. The symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $M^G \vee (M^{\text{op}})^G \subset S^G$ by Proposition 3.3. By Proposition 3.5, the space $L^2(S^G, tr)$ is isomorphic to a sum of bifinite T^G -bimodules indexed by $G_o \backslash G / G_o$.

Suppose that Γ is infinite. Observe that $f : G \rightarrow \mathbb{N}, g \mapsto d(o, go)$ is G_o -bi-invariant map. Moreover, $\text{Im} f = \{m \in \mathbb{N} : \exists w \in V^+, d(o, w) = m\}$ since G acts transitively on V^+ . Therefore, $|G_o \backslash G / G_o| \geq |\text{Im} f|$. Since Γ has infinitely many vertices and is locally finite, there exists vertices in V^+ that are arbitrary far from $o \in V^+$. Hence, $\text{Im} f$ is infinite and $|G_o \backslash G / G_o| = \infty$. This implies that $[S^G : M^G \vee (M^{\text{op}})^G] = [S^G : T^G][T^G : M^G \vee (M^{\text{op}})^G] = \infty$.

Suppose that Γ is finite. Then $|G_o \backslash G / G_o| \leq |G / G_o| = |V^+| < \infty$. Hence, $[S^G : T^G] < \infty$. The group G_o is necessarily finite since Γ has a finite number of vertices and edges and G_o is contained in $\text{Aut}(\Gamma)_o$. As observed in the last proof, the inclusion $M^G \vee (M^{\text{op}})^G \subset T^G$ is isomorphic to $T(o)^{G_o \times G_o} \subset T(o)^{G_o}$ and $T(o)$ is a factor. Hence, $[S^G : M^G \vee (M^{\text{op}})^G] = [S^G : T^G][T^G : M^G \vee (M^{\text{op}})^G] = [S^G : T^G][T(o)^{G_o} : T(o)^{G_o \times G_o}] = [S^G : T^G]|G_o| < \infty$. This concludes the proof. \square

4. DESCRIPTION OF SOME SYMMETRIC ENVELOPING INCLUSIONS VIA HECKE PAIRS

4.1. Crossed products and Hecke pairs.

4.1.1. *The ordinary action case.* Consider an inclusion of groups $H < G$ and assume that G/H is countable. We say that (G, H) is a Hecke pair if the subgroup $H < G$ is almost normal, i.e. for any $g \in G$ the group $H \cap gHg^{-1}$ has finite index inside H and gHg^{-1} . We refer the reader to [1] for more details on Hecke pairs and operator algebras. To (G, H) can be associated a von Neumann algebra $L(G, H)$ analogous to the group von Neumann algebra. In this section, we define the cocycle action of a Hecke pair on a tracial von Neumann algebra and its corresponding twisted crossed product. This has been considered in the framework of ordinary action on C^* -algebras by Palma, see [23, 24]. Our approach is a natural generalization of the construction given in [1, Section 1]. We first define this notion in the ordinary action case which has much simpler formulas.

Let (A, τ) be a finite von Neumann algebra with a normal faithful tracial state. Let $\gamma : G \rightarrow \text{Aut}(A, \tau)$ be a trace-preserving action of G on A . Let $\mathbb{C}[A; G, H]$ be the space of functions $f : G \rightarrow A$ such that $f(hgk) = \gamma_h(f(g))$ for any $g \in G, h, k \in H$, and such that the induced functions $\bar{f} : G/H \rightarrow A$ is finitely supported. Note that if $f \in \mathbb{C}[A; G, H]$ and $g \in G$, then $f(g)$ is fixed by $\sigma(H \cap gHg^{-1})$. We define a multiplication and an involution $*$ on $\mathbb{C}[A; G, H]$ as follows.

$$f_1 f_2(g) = \sum_{s \in \langle G/H \rangle} f_1(s) \gamma_s(f_2(s^{-1}g)), \text{ for any } f_1, f_2 \in \mathbb{C}[A; G, H] \text{ and } g \in G,$$

where $\langle G/H \rangle$ is a system of representatives of G/H ,

$$f^*(g) = \gamma_g(f(g^{-1})^*), \text{ for any } f \in \mathbb{C}[A; G, H] \text{ and } g \in G.$$

The space $\mathbb{C}[A; G, H]$ endowed with those operations is a unital $*$ -algebra. We have an inclusion of A^H inside $\mathbb{C}[A; G, H]$ given by the map $j : a \in A^H \mapsto f_a \in \mathbb{C}[A; G, H]$ such that $f_a(g) = a$ if $g \in H$ and zero otherwise. We identify A^H and $j(A^H)$. Consider the linear functional $\omega : \mathbb{C}[A; G, H] \rightarrow \mathbb{C}, f \mapsto \tau(f(1))$. Let $\pi = \gamma \otimes \text{Ad}(\lambda) : G \rightarrow \text{Aut}(A \bar{\otimes} B(\ell^2(G/H)))$ be the group action defined as $\pi_g(a \otimes e_{h,k}) = \gamma_g(a) \otimes e_{gh, gk}$, for any $g \in G, h, k \in G/H$ and $a \in A$. Denote by $\text{Tr} : B(\ell^2(G/H))_+ \rightarrow \mathbb{R}_+$ the usual normal faithful semi-finite tracial weight on $B(\ell^2(G/H))$ that sends any minimal projection to 1.

Proposition 4.1. *The linear functional ω is faithful and the $*$ -algebra $\mathbb{C}[A; G, H]$ acts by bounded operators on $L^2(\mathbb{C}[A; G, H], \omega)$ by left multiplication. Let $vN[A; G, H]$ be the von Neumann algebra generated by this left action that we call the crossed product of A by (G, H) . We have an isomorphism φ of von Neumann algebras from $vN[A; G, H]$ onto the fixed point space $(A \bar{\otimes} B(\ell^2(G/H)))^G$ such that $\varphi(f) = \sum_{s, t \in \langle G/H \rangle} \gamma_s(f(t)) \otimes e_{s, st}$ for any $f \in \mathbb{C}[A; G, H]$. In particular,*

$$\varphi(A^H) = \left\{ \sum_{s \in \langle G/H \rangle} \gamma_s(a) \otimes e_{s, s} : a \in A^H \right\}.$$

Proof. If $f \in \mathbb{C}[A; G, H]$, then $\omega(ff^*) = \tau(ff^*(1)) = \sum_{s \in \langle G/H \rangle} \tau(f(s)f(s)^*)$. Hence, if $\omega(ff^*) = 0$, then $f(s) = 0$ for any $s \in \langle G/H \rangle$ since τ is faithful. This implies that ω is faithful.

Consider the Hilbert space $\ell^2(G/H, A)$ of right- H -invariant functions $f : G \rightarrow L^2(A)$ such that $\sum_{s \in \langle G/H \rangle} \tau(f(s)f(s)^*) < \infty$ with the norm $\|f\|_2 = \sqrt{\sum_{s \in \langle G/H \rangle} \tau(f(s)f(s)^*)}$.

The Hilbert space $L^2(\mathbb{C}[A; G, H], \omega)$ is isometric to a subspace of $\ell^2(G/H, A)$. We identify $L^2(\mathbb{C}[A; G, H], \omega)$ and this isometric subspace. Consider $r \in \langle G/H \rangle$ and $\{h_1, \dots, h_k\} \subset H$ such that HrH is equal to the disjoint union of the right cosets $\bigcup_{i=1}^k h_i r H$. Consider $a \in A$ and the function $f \in \mathbb{C}[A; G, H]$ such that its support is contained in HrH and $f(r) = a$. Consider an element $\xi \in \mathbb{C}[A; G, H] \subset L^2(\mathbb{C}[A; G, H], \omega)$. We have that

$$(f\xi)(g) = \sum_{s \in \langle G/H \rangle} f(s) \gamma_s(\xi(s^{-1}g)) = \sum_{i=1}^k \gamma_{h_i}(a) \gamma_{h_i r}(\xi((h_i r)^{-1}g)), \quad g \in G.$$

We fix $1 \leq i \leq k$. Consider the map $\eta_i : g \in G \mapsto \gamma_{h_i}(a) \gamma_{h_i r}(\xi((h_i r)^{-1}g))$. Note that $\eta_i \in \ell^2(G/H, A)$ and $f\xi = \sum_{j=1}^k \eta_j$. We have that

$$\begin{aligned} \|\eta_i\|_2^2 &= \sum_{s \in \langle G/H \rangle} \tau(\gamma_{h_i}(a) \gamma_{h_i r}(\xi((h_i r)^{-1}s) \gamma_{h_i r}(\xi((h_i r)^{-1}s)^* \gamma_{h_i}(a)^*)) \\ &= \sum_{s \in \langle G/H \rangle} \tau(a^* a \gamma_r(\xi((h_i r)^{-1}s) \xi((h_i r)^{-1}s)^*)) \\ &\leq \|a\|_A^2 \sum_{s \in \langle G/H \rangle} \tau(\xi((h_i r)^{-1}s) \xi((h_i r)^{-1}s)^*) = \|a\|_A^2 \|\xi\|_2^2, \end{aligned}$$

where $\|\cdot\|_A$ is the C^* -norm of A . Hence, $\|f\xi\|_2 \leq k\|a\|_A\|\xi\|_2$. Since the subspace $\mathbb{C}[A; G, H] \subset L^2(\mathbb{C}[A; G, H], \omega)$ is dense, we obtain that the left multiplication by f on $L^2(\mathbb{C}[A; G, H], \omega)$ is bounded. Therefore, the left multiplication of $\mathbb{C}[A; G, H]$ defines a $*$ -representation of $\mathbb{C}[A; G, H]$ on the GNS Hilbert space $L^2(\mathbb{C}[A; G, H], \omega)$. It is clear the φ defines a normal $*$ -morphism from $\text{vN}[A; G, H]$ to the fixed point space $(A \bar{\otimes} B(\ell^2(G/H)))^G$. If $\varphi(f) = 0$, then $\gamma_s(f(t)) = 0$ for any $s, t \in \langle G/H \rangle$. Hence, $f = 0$ and φ is injective. Consider $x = \sum_{s,t \in G/H} x_{s,t} \otimes e_{s,t} \in (A \bar{\otimes} B(\ell^2(G/H)))^G$. Since x is G -invariant, we have that $x_{gs,gt} = \gamma_g(x_{s,t})$ for any $g \in G, s, t \in G/H$. Therefore, $x = \varphi(f)$ where $f(t) = x_{1,t}, t \in \langle G/H \rangle$. Hence, φ is surjective. We obtain that φ realizes an isomorphism of von Neumann algebras. The last assertion of the proposition follows from the definition of φ . \square

4.1.2. *The twisted case.* Let G, H, A, τ be as above. A cocycle action of the Hecke pair (G, H) on the tracial von Neumann algebra (A, τ) is a couple of maps (γ, u) where

$$\gamma : G \times G/H \longrightarrow \text{Aut}(A, \tau) \text{ and } u : G \times G/H \times G/H \longrightarrow U(A)$$

satisfying the following axioms:

- (1) $\gamma_{1,s} = \text{Id}$,
- (2) $\gamma_{gh,s} = \gamma_{g,hs} \circ \gamma_{h,s}$,
- (3) $\gamma_{g,s} = \text{Ad}(u_{g,s,t}) \circ \gamma_{g,t}$,
- (4) $u_{1,s,t} = u_{g,s,s} = 1$,
- (5) $u_{g,s,t} u_{g,t,r} = u_{g,s,r}$, and
- (6) $u_{gh,s,t} = \gamma_{g,hs}(u_{h,s,t}) u_{g,hs,ht}$ for any $g, h \in G$ and $s, t, r \in G/H$.

We continue to denote by $\mathbb{C}[A; G, H]$ the space of functions $f : G \longrightarrow A$ such that $f(hgk) = \gamma_{h,1}(f(g))u_{h,1,g}$ for any $g \in G, h, k \in H$, and such that the induced functions $\bar{f} : G/H \longrightarrow A$ is finitely supported. We define a multiplication and an involution $*$ on

$\mathbb{C}[A; G, H]$ as follows.

$$f_1 f_2(g) = \sum_{s \in \langle G/H \rangle} f_1(s) \gamma_{s,1}(f_2(s^{-1}g)) u_{s,1,s^{-1}g}, \text{ for any } f_1, f_2 \in \mathbb{C}[A; G, H] \text{ and } g \in G,$$

where $\langle G/H \rangle$ is a system of representatives of G/H ,

$$f^*(g) = \gamma_{g,g^{-1}}(f(g^{-1})^*) u_{g,g^{-1},1}, \text{ for any } f \in \mathbb{C}[A; G, H] \text{ and } g \in G.$$

A careful check shows that the space $\mathbb{C}[A; G, H]$ endowed with those operations is a unital associative $*$ -algebra. Observe that the map $h \in H \mapsto \gamma_{h,1} \in \text{Aut}(A, \tau)$ is a group morphism. Hence, we have an ordinary action of H on A . Denote by A^H the algebra of $a \in A$ such that $\gamma_{h,1}(a) = a$ for any $h \in H$. We have an inclusion of A^H inside $\mathbb{C}[A; G, H]$ given by the map $j : a \in A^H \mapsto f_a \in \mathbb{C}[A; G, H]$ such that $f_a(g) = a$ if $g \in H$ and zero otherwise. Consider the linear functional $\omega : \mathbb{C}[A; G, H] \rightarrow \mathbb{C}, f \mapsto \tau(f(1))$.

Proposition 4.2. *The linear functional ω is faithful and the $*$ -algebra $\mathbb{C}[A; G, H]$ acts by bounded operators on $L^2(\mathbb{C}[A; G, H], \omega)$ by left multiplication. Let $\text{vN}[A; G, H]$ be the von Neumann algebra generated by this left action that we call the twisted crossed product of A by (G, H) . There exists a unique group morphism $\pi : G \rightarrow \text{Aut}(A \bar{\otimes} B(\ell^2(G/H)))$ satisfying that $\pi_g(a \otimes e_{s,t}) = \gamma_{g,s}(a) u_{g,s,t} \otimes e_{gs,gt}$ for any $a \in A, g \in G, s, t \in G/H$. We have an isomorphism φ of von Neumann algebras from $\text{vN}[A; G, H]$ onto the fixed point space $(A \bar{\otimes} B(\ell^2(G/H)))^G$ such that $\varphi(f) = \sum_{s,t \in \langle G/H \rangle} \gamma_{s,1}(f(t)) u_{s,1,t} \otimes e_{s,st}$ for any $f \in \mathbb{C}[A; G, H]$. In particular,*

$$\varphi(A^H) = \left\{ \sum_{s \in \langle G/H \rangle} \gamma_{s,1}(a) \otimes e_{s,s} : a \in A^H \right\}.$$

Proof. The proof is similar to the proof of Proposition 4.1. \square

Remark 4.3. If $A = \mathbb{C}$, then the von Neumann algebra $\text{vN}[A; G, H]$ is the von Neumann algebra $L(G, H)$ (possibly twisted by a cocycle) associated to the Hecke pair (G, H) [1]. If $H < G$ is a normal subgroup, then the von Neumann algebra $\text{vN}[A; G, H]$ is isomorphic to a (twisted) crossed product $A^H \rtimes (G/H)$ of the fixed point space A^H by the quotient group G/H . If the standard system of matrix units of $B(\ell^2(G/H))$ is G -invariant, i.e. $\pi_g(1 \otimes e_{s,t}) = 1 \otimes e_{gs,gt}$ for any $g \in G, s, t \in G/H$, then $(A \bar{\otimes} B(\ell^2(G/H)))^G$ is isomorphic to $\text{vN}[A; G, H]$ for an ordinary action of (G, H) on A .

4.2. Description of symmetric enveloping inclusions. Let (Γ, μ) be a weighted graph with vertex set $V = V^+ \cup V^-$ and let \mathcal{P} be its associated bipartite graph planar algebra. Moreover, assume that μ is constant on the set of edges with source in V^+ . Let us fix a vertex $o \in V^+$ and denote by $T \subset S$ the symmetric enveloping inclusion of \mathcal{P} . The set $\{e_v : v \in V^+\}$ is the set of minimal projections of $\ell^\infty(V^+)$ and $\{e_{v,w} : v, w \in V^+\}$ is the standard system of matrix units of $B(\ell^2(V^+))$. We continue to denote by $T(v)$ the corner Tp_v for any $v \in V^+$.

Theorem 4.4. *The symmetric enveloping inclusion of \mathcal{P} is isomorphic to*

$$T(o) \bar{\otimes} \ell^\infty(V^+) \subset T(o) \bar{\otimes} B(\ell^2(V^+)),$$

where the inclusion is given by $a \otimes e_v \mapsto a \otimes e_{v,v}$ for any $a \in T(o), v \in V^+$.

Proof. By Proposition 2.2, $Tr(p_v) = Tr(p_o)$ for any $v \in V^+$. Since S is a factor, there exists a system of matrix units $\{\epsilon_{v,w} : v, w \in V^+\} \subset S$ such that $\epsilon_{v,w}\epsilon_{v,w}^* = p_v$ and $\epsilon_{v,w}^*\epsilon_{v,w} = p_w$ for any $v, w \in V^+$. Consider $v \in V^+$ and the map $\beta_v : T(o) \longrightarrow S, x \longmapsto \epsilon_{v,o}x\epsilon_{o,v}$. Since $p_vSp_v = T(v)$, we have that the range of β_v is contained in $T(v)$. Observe that β_v realizes an isomorphism of von Neumann algebras from $T(o)$ onto $T(v)$ and its inverse is the map $\beta_v^{-1} : T(v) \longrightarrow T(o), x \longmapsto \epsilon_{o,v}x\epsilon_{v,o}$. Observe that S is the weak closure of the vector space $\text{Span}(x\epsilon_{v,w} : v, w \in V^+, x \in T(v))$. Consider the densely defined map $\phi : S \longrightarrow T(o) \bar{\otimes} B(\ell^2(V^+))$ defined as $\phi(x\epsilon_{v,w}) = \beta_v^{-1}(x) \otimes e_{v,w}$ for any $v, w \in V^+, x \in T(v)$. This map is clearly a $*$ -morphism and extends to a normal map from S to $T(o) \bar{\otimes} B(\ell^2(V^+))$. Since S is a factor and ϕ is not identically equal to zero we have that ϕ is injective. The range of ϕ is weakly dense since it contains the set $\{x \otimes e_{v,w} : x \in T(o), v, w \in V^+\}$. This implies that ϕ is surjective. Therefore, ϕ realizes an isomorphism of von Neumann algebras. If $x \in T$, then $\phi(x) = \sum_{v \in V^+} \beta_v^{-1}(xp_v) \otimes e_{v,v}$. This implies that $\phi(T)$ is contained in $T(o) \bar{\otimes} \ell^\infty(V^+)$. Furthermore, $\phi(T)$ is clearly dense in $T(o) \bar{\otimes} \ell^\infty(V^+)$. Therefore, $\phi(T) = T(o) \bar{\otimes} \ell^\infty(V^+)$. \square

Consider a subgroup $G < \text{Aut}(\mathcal{P})$ such that $\mathcal{Q} := \mathcal{P}^G$ is a subfactor planar algebra. In particular, G acts transitively on V^+ . We identify V^+ and G/G_o . Let $\sigma : G \longrightarrow \text{Aut}(S)$ be the action defined in Section 3. Note that we have an action of $G_o \times G_o$ on the corner $T(o)$ given by $(g, h) \cdot a \otimes b^{\text{op}} = \sigma_g(a) \otimes \sigma_h(b)^{\text{op}}$ for any $g, h \in G_o, a, b \in M(o)$. We denote by $T(o)^{G_o \times G_o}$ the fixed point space with respect to this action. Consider the diagonal inclusion of groups $g \in G_o \longmapsto (g, g) \in G_o \times G_o$ and the corresponding inclusion of von Neumann algebras $T(o)^{G_o \times G_o} \subset T(o)^{G_o}$. Denote by $\mathcal{Q} = \mathcal{P}^G$ the fixed point space of \mathcal{P} with respect to the action of G .

Theorem 4.5. *There exists a cocycle action (γ, u) of the Hecke pair (G, G_o) on the II_1 factor $T(o)$ such that $\gamma_{g,1}(a \otimes b^{\text{op}}) = \sigma_g(a) \otimes \sigma_g(b)^{\text{op}}$ for any $g \in G, a, b \in M(o)$, and such that the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to*

$$T(o)^{G_o \times G_o} \subset vN[T(o); G, G_o].$$

Proof. The map $g \in G \longmapsto go \in V^+$ is surjective since the action is transitive. It realizes a bijection from G/G_o onto V^+ . Denote by $\langle G/G_o \rangle$ a system of representatives of G/G_o . By Theorem 4.4, the symmetric enveloping inclusion of \mathcal{P} is isomorphic to $T(o) \bar{\otimes} \ell^\infty(V^+) \subset T(o) \bar{\otimes} B(\ell^2(V^+))$ via the map $\phi : S \longrightarrow T(o) \bar{\otimes} B(\ell^2(V^+))$ that satisfies $\phi(a\epsilon_{v,w}) = \epsilon_{o,v}a\epsilon_{v,o} \otimes e_{v,w}$ for any $v, w \in V^+, a \in T(v)$. We have that the group action

$$\pi : G \longrightarrow \text{Aut}(T(o) \bar{\otimes} B(\ell^2(V^+))), g \longmapsto \phi \circ \sigma_g \circ \phi^{-1}$$

satisfies that

$$(11) \quad \pi_g(T(o) \otimes e_{v,w}) = T(o) \otimes e_{gv,gw} \text{ and } \pi_g(1 \otimes e_{v,v}) = 1 \otimes e_{gv,gv} \text{ for any } g \in G, v, w \in V^+.$$

This implies that for any $g \in G, v, w \in V^+$ there exists a unitary $u_{g,v,w} \in U(T(o))$ such that $\pi_g(1 \otimes e_{v,w}) = u_{g,v,w} \otimes e_{gv,gw}$. Furthermore, for any $a \in T(o), v \in V^+, g \in G$ there exists $\gamma_{g,v}(a) \in T(o)$ such that $\pi_g(a \otimes e_{v,v}) = \gamma_{g,v}(a) \otimes e_{gv,gv}$. Since π_g is an automorphism and $\pi_g(T(o) \otimes e_{v,v}) = T(o) \otimes e_{gv,gv}$ we necessarily have that $\gamma_{g,v}$ is an automorphism of $T(o)$ for any $g \in G, v \in V^+$. From (11) and the fact that π is a group action we can deduce that (γ, u) is a cocycle action of (G, G_o) on the II_1 factor $T(o)$. Let $vN[T(o); G, G_o]$ be the crossed product von Neumann algebra associated to the cocycle action (γ, u) of the

Hecke pair (G, G_o) on the II_1 factor $T(o)$. By Proposition 4.2, we obtain that the inclusion $T(o)^{G_o} \subset \text{vN}[T(o); G, G_o]$ is isomorphic to the inclusion

$$(12) \quad \left\{ \sum_{s \in \langle G/G_o \rangle} \sigma_s(x) \otimes e_{so,so} : x \in T(o)^{G_o} \right\} \subset (T(o) \bar{\otimes} B(\ell^2(V^+)))^G,$$

where $(T(o) \bar{\otimes} B(\ell^2(V^+)))^G$ is the fixed point space under the action of π . Consider the isomorphism $\phi : S \rightarrow T(o) \bar{\otimes} B(\ell^2(V^+))$ of the proof of Theorem 4.4. The symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $M^G \vee (M^{\text{op}})^G \subset S^G$ by Proposition 3.3. We have that $\phi(S^G) = (T(o) \bar{\otimes} B(\ell^2(V^+)))^G$. Observe that $\phi(M^G) = \{\sum_{s \in \langle G/G_o \rangle} \sigma_s(a) \otimes 1 \otimes e_{so,so} : a \in M(o)^{G_o}\}$ and $\phi((M^{\text{op}})^G) = \{\sum_{s \in \langle G/G_o \rangle} 1 \otimes (\sigma_s(b))^{\text{op}} \otimes e_{so,so} : b \in M(o)^{G_o}\}$. This last observation together with the characterization of the image of $T(o)^{G_o}$ given in (12) imply that the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to the inclusion $T(o)^{G_o \times G_o} \subset \text{vN}[T(o); G, G_o]$. \square

4.3. Examples. We present examples of subfactor planar algebras for which we can describe their symmetric enveloping inclusion via our last theorem.

4.3.1. Diagonal subfactors. The symmetric enveloping inclusion of a diagonal subfactor is known to be isomorphic to the (twisted) crossed product of a II_1 factor by a group [30, Section 3]. We give here a new proof of this fact which is a particular case of Theorem 4.5. Consider a II_1 factor L and a finite set $\{g_1, \dots, g_n\}$ of outer automorphisms of L . Denote by G the subgroup of the outer automorphism group $\text{Out}(L)$ generated by $\{g_1, \dots, g_n\}$. Let $N \subset M$ be the subfactor equal to the inclusion $\{\sum_{i=1}^{n+1} \alpha_{g_i}(x) \otimes e_{i,i} : x \in L\} \subset L \otimes \mathcal{M}_{n+1}(\mathbb{C})$, where $\{e_{i,j} : i, j = 1, \dots, n+1\}$ is the usual system of matrix units of the type I_{n+1} factor $\mathcal{M}_{n+1}(\mathbb{C})$ and $g_{n+1} = 1$. This subfactor is called a diagonal subfactor. By [3, 7], the subfactor planar algebra of $N \subset M$ can be described as the fixed point planar algebra of a bipartite graph planar algebra as follows. Consider the bipartite graph Γ , where $V^+ = \{v_g^+ : g \in G\}$ is a copy of G and $V^- = \{v_g^- : g \in G\}$ is another copy of G . For any $g \in G$ and $1 \leq i \leq n+1$ there is an edge from v_g^+ to $v_{gg_i}^-$. Let μ be the weight of Γ that assigns 1 to any edge and let \mathcal{P} be the bipartite graph planar algebra associated to (Γ, μ) . The group G acts on the weighted graph (Γ, μ) by left multiplication. The fixed point planar algebra $\mathcal{Q} = \mathcal{P}^G$ is isomorphic to the subfactor planar algebra of $N \subset M$. For any even vertex of Γ , the subgroup of G that fixes this vertex is trivial. Therefore, by Theorem 4.5 and Remark 4.3, there exists a II_1 factor A and a trace-preserving cocycle action of G on A such that the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $A \subset A \rtimes G$.

4.3.2. Bisch-Haagerup subfactors. Consider a II_1 factor L and two finite subgroups $H, K \subset \text{Out}(L)$. Assume that the intersection $H \cap K$ is the trivial group. Denote by G the subgroup of $\text{Out}(L)$ generated by H and K . Assume that there exists a group morphism $\alpha : G \rightarrow \text{Aut}(L)$ which is a lift of the inclusion $G \subset \text{Out}(L)$. The subfactor $L^K \subset L \rtimes H$ is called a Bisch-Haagerup-subfactor [5]. By [4, 7], its subfactor planar algebra is isomorphic to the fixed point space of a bipartite graph planar algebra as follows. Consider the bipartite graph Γ where $V^+ = \{v_{gH} : gH \in G/H\}$ is a copy of the coset space G/H and $V^- = \{v_{gK} : gK \in G/K\}$ is a copy of the coset space G/K . The set of edges $C_1 = \{e_g : g \in G\}$ is equal to a copy of G where e_g is an edge between v_{gH} and v_{gK} . Let μ be the weight on Γ that assigns $|K|^{1/2}/|H|^{1/2}$ to any edge in C_1^+ . Let \mathcal{P} be the bipartite graph planar algebra associated to (Γ, μ) . The group G acts by left multiplication on

(Γ, μ) and the fixed point space $\mathcal{Q} = \mathcal{P}^G$ is isomorphic to the subfactor planar algebra of the subfactor $L^K \subset L \rtimes H$. Consider the even vertex v_H . The subgroup of G that fixes v_H is equal to H . By Theorem 4.5, there exists a II_1 factor A and a cocycle action of the Hecke pair (G, H) on $A \bar{\otimes} A^{\text{op}}$ such that H acts on A and such that the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $A^H \bar{\otimes} (A^{\text{op}})^H \subset \text{vN}[A \bar{\otimes} A^{\text{op}}; G, H]$.

4.3.3. Binary trees and subfactors. Consider two non-zero natural numbers r^+ and r^- . Let $\Gamma = T(r^\pm)$ be the infinite bipartite tree where any vertex $v \in V^\pm$ has degree r^\pm . Define the weight μ such that $\mu(a) = \sqrt{r^-/r^+}$ for any $a \in C_1^+$. We have that the weighted graph (Γ, μ) satisfies all the assumptions of Section 1 and has modulus δ equal to $\sqrt{r^+r^-}$. Let \mathcal{P} be the bipartite graph planar algebra associated to (Γ, μ) . The group $\text{Aut}(\Gamma, \mu)$ is equal to the automorphism group of the bipartite graph $G = \text{Aut}(\Gamma)$ which acts transitively on the even and odd vertices and on the edges. Let o be an even vertex of Γ and G_o the subgroup of G that fixes this vertex. The group G_o is an infinite group. Let \mathcal{Q} be the fixed point planar algebra under the action of G . It is an irreducible subfactor planar algebra. By Theorem 4.5, there exists a II_1 factor A and a cocycle action of the Hecke pair (G, G_o) on $A \bar{\otimes} A^{\text{op}}$ such that G_o acts on A and such that the symmetric enveloping inclusion of \mathcal{Q} is isomorphic to $A^{G_o} \bar{\otimes} (A^{\text{op}})^{G_o} \subset \text{vN}[A \bar{\otimes} A^{\text{op}}; G, G_o]$.

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